

Math 230 Section BE1/BB1 Introduction to Differential Equations

Definition 1. An (*ordinary*) *differential equation (DE)* is an equation involving an unknown function (or functions), one or more of its derivatives, and possibly the independent variable.

Examples: $y' = 4y$, $y' = 1 + \cos x$, $e^{y \cdot y'} + x^2 - y'' = 17$

A *solution* to a DE is a function which makes the equation true.

Example 1. Is $y(x) = 3e^{4x}$ a solution to $y' = 4y$?

Answer 1. Yes, since $y' = 12e^{2x} = 4(3e^{2x}) = 4y$.

Example 2. Is $y(x) = \sin x$ a solution to $y' = 1 + \cos x$? Find all solutions to $y' = 1 + \cos x$.

Answer 2. No, since $y' = \cos x \neq 1 + \cos x$.

To find all solutions to this differential equation, we integrate both sides:

$$y(x) = \int y'(x) dx = \int (1 + \cos x) dx = x + \sin x + C,$$

where C is an arbitrary constant. Every solution to the equation takes this form for some value of C .

Example 3. Find all solutions to the differential equation $y' = 4y$.

Answer 3. After seeing the answer to the first example, it's not too hard to guess that $y(x) = Ce^{4x}$ is a solution for each constant C . To see that these are all solutions, we rewrite the equation in the form

$$4 = \frac{y'}{y} = (\ln |y|)'$$

(using the Chain Rule). Then we can integrate each side:

$$4x + C' = \ln |y|$$

and solve for y : $y(x) = \pm e^{4x+C'} = \pm e^{C'} e^{4x}$. Here C' denotes an arbitrary constant of integration. As C' varies over all real numbers, the expression $\pm e^{C'}$ varies over all real numbers. To simplify the final answer, we write $C = \pm e^{C'}$; then the general solution to the differential equation is

$$y(x) = Ce^{4x}.$$

(This is a simple case of the method of *separation of variables* which can be used to solve some first-order DE's. We will learn this method in section 7.4.)

The *order* of a differential equation is the highest number of derivatives which occurs in any term. Both $y' = 4y$ and $y' = 1 + \cos x$ are *first-order* equations. Notice that the general solutions to these equations each involved *one* arbitrary constant. This is not a coincidence; typically the general solution to an n th order differential equation involves n arbitrary constants. (For example, this is true for *linear* equations; both of our examples are linear.)

Often we include together with the differential equation an *initial value*, usually the value of the unknown function at a particular input. The combination of the differential equation and the initial value is called an *initial value problem (IVP)*. Typically, an IVP will have a unique solution (called a *particular solution*), which can be determined by (i) finding the general solution to the DE, and (ii) using the initial value to solve for the arbitrary constant(s).

Example 4. Solve the IVP $y' = 4y$, $y(0) = 5$.

Answer 4. As discussed before, the general solution is $y(x) = Ce^{4x}$. We compute $5 = y(0) = Ce^0 = C$, so the particular solution is $y(x) = 5e^{4x}$.

Example 5. Solve the IVP $y' = 1 + \cos x$, $y(\pi) = 2\pi$.

Answer 5. As discussed before, the general solution is $y(x) = x + \sin x + C$. We compute $2\pi = y(\pi) = \pi + \sin \pi + C = \pi + C$, so $C = \pi$ and the particular solution is $y(x) = x + \sin x + \pi$.

Higher-order DE's require additional initial values.

Example 6. Solve the IVP $y'' = -g$, $y(0) = y_0$, $y'(0) = v_0$ for $y(t)$.

This DE models *free fall*, i.e., the motion of a falling body subject only to the force of gravity. The quantity g represents the gravitational constant, $g = 32$ ft/sec/sec or $g = 9.8$ m/sec/sec; while y_0 represents the *initial height* and v_0 represents the *initial velocity* of the object. We use t for the independent variable to denote time.

Answer 6. Integrating once gives $y' = -gt + C_1$. The first initial condition gives $v_0 = y'(0) = -0 + C_1 = C_1$ so $y' = -gt + v_0$. Integrating again gives $y = -\frac{1}{2}gt^2 + v_0t + C_2$. The second initial condition gives $y_0 = y(0) = -0 + 0 + C_2$ so

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0.$$

Vector Fields

Vector fields (also known as slope fields) provide a way to visualize first-order differential equations. A first-order DE $y' = F(x, y)$ can be rule assigning to each point (x_0, y_0) an infinitesimal line segment with slope $F(x_0, y_0)$. Assuming that there is a unique solution $y = f(x)$ to the IVP $y' = F(x, y)$, $y(x_0) = y_0$, it follows that the slope of the tangent line to $y = f(x)$ at $x = x_0$ must be equal to $F(x, y)$. Plotting short line segments in the xy -plane with the appropriate slopes generates the *vector field* associated with this differential equation.

Example 7. Consider the first-order differential equation $y' = x - y + 1$. At each point (x, y) we plot a short line segment whose slope is equal to the value of $x - y + 1$. The figure on the left shows the result.

In the figure on the right, we have plotted three solution curves for this differential equation. Notice the relationship between each of these curves and the vector field; at each point $(x, y(x))$ on one of the solution curves, the tangent vector to the solution curve coincides with the short line segment which we drew.

One way to think about this is that the vector field gives “instructions” telling the solution curve the direction in which it should travel (for an infinitesimal length of time). This point of view is closely related to Euler’s method for approximating solutions to differential equations, which we will discuss in section 6.3.

Notice that the vector field gives some interesting qualitative information about the solution curves for the differential equation. For example, it suggests that there is a straight line (linear) solution. Can you find a solution to $y' = x - y + 1$ of the form $y = mx + b$ for some m and b ? Does your answer match the vector field?

In-class Group Work

Exercise 1. Is $y(x) = x + 3e^{-2x}$ a solution to the DE $y' + 2y = 1 + 2x$?

Exercise 2. Find all solutions to the DE $y' = x^2 + 4x + 3$.

Exercise 3. (a) Find the unique solution to the IVP $y' = 6x^2 + 2x + 1$, $y(0) = 5$.

(b) Find the unique solution to the IVP $y' = 6x^2 + 2x + 1$, $y(1) = 5$.

Exercise 4. A man standing on a platform throws a ball directly upward from an initial height of 24 ft at an initial velocity of 40 ft/sec. Find a formula for $y(t)$, the height of the ball above the ground after t seconds. How high does the ball go? When does it hit the ground?

Exercise 5. You open a savings account with \$ 1000 and make monthly contributions of \$ 100 thereafter. The interest rate on the account is 5 % per year. (For simplicity, assume that the monthly contributions are added to the account continuously over each month, and that the interest is compounded continuously.) Write an IVP (differential equation plus initial value condition) which models $A(t)$, the amount of money in the account after t years.

Exercise 6. Do exercise 3 of section 4.1.

Exercise 7. Match the vector fields in exercises 7–10 of section 4.1 with one of the corresponding differential equations labelled (i)–(x).

Homework (due Monday 2/21)

Exercise 1. Find the unique solution to the IVP $y' = \frac{1}{x^2}$, $y(2) = \frac{5}{2}$.

Exercise 2. The rate at which an object cools is proportional to the difference between the current temperature T of the object and the ambient (room) temperature A (a constant). Write a differential equation for the temperature T of the object as a function of time t .

Exercise 3. Match the vector fields in exercises 11–14 of section 4.1 with one of the corresponding differential equations labelled (i)–(x).

Exercise 4. Do exercise 31 of section 4.1.

Exercise 5. Do exercise 4 of section 6.3.

Exercise 6. Do exercise 9(a),(b),(c) of section 6.3.