

Math 230 Section BE1 Review Sheet for Exam IV

The fourth midterm exam will take place on **Friday, April 22** in class. It will cover the material in Chapter 11: sequences and series, tests for convergence and divergence of series, absolute and conditional convergence, power series and Taylor series.

- Make sure that you understand the difference between a sequence and a series!¹ Every series

$$\sum_{n=0}^{\infty} a_n$$

of real numbers has two sequences associated to it: the sequence (a_n) of *terms* in the series, and the sequence (S_n) of *partial sums* of the series, defined by

$$S_n = \sum_{k=0}^n a_k.$$

The series *converges* or *diverges* if and only if the sequence of partial sums *converges* or *diverges*. If the series converges, the *sum* of the series is the limit of the partial sums.

A series $\sum_{n=0}^{\infty} a_n$ *converges absolutely* if the series $\sum_{n=0}^{\infty} |a_n|$ converges. A series $\sum_{n=0}^{\infty} a_n$ *converges conditionally* if it converges, but the series $\sum_{n=0}^{\infty} |a_n|$ diverges.

- You should know in which cases the basic comparison series (harmonic series, p -series, geometric series) converge or diverge.
- You should know the statements of the Ratio Test, the Integral Test, and the Alternating Series Test, as well as the Taylor series theorem from section 11.7. *I will ask you to state one of these four theorems or tests (including all relevant hypotheses and conclusions) on the exam!* You should be able to apply the tests to decide whether a given series converges absolutely, converges conditionally, or diverges. You should be able to estimate the value of a given infinite sum by “bounding the tail”.

¹Major League Baseball’s *World Series* is actually a sequence ...

- You should be able to find the radius and interval of convergence of a power series. (Don’t forget to check the endpoints!) You should be able to manipulate power series algebraically and analytically (add them, multiply them, differentiate them, integrate them, etc.)
- You should know how to find the coefficients in the Taylor series for a given function.

Practice Exercises

1. Consider the sequence $a_n = \frac{n^3}{2^n}$.
 - (a) Find, with proof, a number N so that $a_{n+1} \leq a_n$ for all $n \geq N$.
 - (b) Why does part (a) imply that $\lim_{n \rightarrow \infty} a_n$ exists?
 - (c) Evaluate, with proof, $\lim_{n \rightarrow \infty} a_n$.

2. Find $\lim_{n \rightarrow \infty} a_n$ for $a_n = \sqrt{n^4 + 4n^2} - n^2$.

3. Is there a convergent series $\sum_{k=1}^{\infty} a_k$ for which $\sum_{k=1}^{\infty} \frac{1}{\ln a_k}$ diverges? If so, give an example.

4. Find the interval and radius of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^4 2^n}.$$

5. The power series for $\sin(x)$ and $\cos(x)$ are

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots$$

and

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$$

Verify the Pythagorean identity by showing that the power series for $\sin^2(x) + \cos^2(x)$ begins with 1 and contains no terms involving powers of x up to x^6 .

6. Find the Taylor series for $f(x) = e^{2x}$ at $x = 1$.
7. Find the function represented by the power series

$$\sum_{k=2}^{\infty} k(k-1)x^{k-2}.$$

Solutions to Practice Exercises

1. (a) $N = 4$ works. If $n \geq 4$, then

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(n+1)^3}{2^{n+1}} \cdot \frac{2^n}{n^3} \\ &= \frac{(n+1)^3}{2n^3} \\ &= \frac{1}{2} \left(1 + \frac{1}{n}\right)^3 \\ &\leq \frac{1}{2}(1.25)^3 \quad (\text{Why?}) \\ &= 0.9765 \\ &< 1. \end{aligned}$$

(b) The sequence $(a_n)_{n=4}^\infty$ is a decreasing sequence of numbers bounded below by zero. By the Monotone Convergence Test for sequences, $\lim_{n \rightarrow \infty} a_n$ exists.

(c) $\lim_{n \rightarrow \infty} a_n = 0$ From our proof to part (a) we see that $a_{n+1} \leq 0.99a_n$ whenever $n \geq 4$. Thus

$$a_5 \leq 0.99a_4, \quad a_6 \leq (0.99)^2 a_4, \quad a_7 \leq (0.99)^3 a_4, \dots$$

and in general,

$$a_n \leq (0.99)^{n-4} a_4 = 4(0.99)^{n-4} \leq 5(0.99)^n$$

for $n \geq 4$. By the squeeze theorem for limits,

$$0 \leq \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} 5(0.99)^n = 0$$

since $0.99 < 1$, so $\lim_{n \rightarrow \infty} a_n = 0$.

2. The value of the limit is $\boxed{2}$. Rationalizing the denominator,

$$\begin{aligned} a_n &= \frac{\sqrt{n^4 + 4n^2} - n^2}{1} \cdot \frac{\sqrt{n^4 + 4n^2} + n^2}{\sqrt{n^4 + 4n^2} + n^2} \\ &= \frac{4n^2}{\sqrt{n^4 + 4n^2} + n^2} = \frac{4}{\sqrt{1 + \frac{4}{n^2}} + 1}, \end{aligned}$$

so $\lim_{n \rightarrow \infty} a_n = \frac{4}{2} = 2$.

3. One such series has terms $\boxed{a_k = e^{-k}}$. Indeed, $\sum_{k=1}^\infty a_k = \sum_{k=1}^\infty (e^{-1})^k$ converges since $e^{-1} < 1$, but

$$\sum_{k=1}^\infty \frac{1}{\ln a_k} = \sum_{k=1}^\infty \frac{1}{\ln(e^{-k})} = -\sum_{k=1}^\infty \frac{1}{k}$$

is the (negative of the) harmonic series, hence diverges.

4. The series converges on the closed interval $\boxed{[3, 7]}$. The Ratio Test gives

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{(x-5)^{n+1}}{(n+1)^2 2^{n+1}} \cdot \frac{n^4 2^n}{(x-5)^n} \right| &= \lim_{n \rightarrow \infty} \frac{n^4}{(n+1)^4} \left| \frac{x-5}{2} \right| \\ &= \left| \frac{x-5}{2} \right|, \end{aligned}$$

so the convergence interval is centered at $x = 5$ and has radius equal to 2. Both endpoints are included in this interval, since the series

$$\sum_{n=1}^\infty \frac{(-1)^n}{n^4}$$

converges absolutely (by the p -series test).

5. Squaring the power series for sine and cosine gives

$$\sin^2(x) = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \dots$$

and

$$\cos^2(x) = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \dots,$$

so

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= \left(x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \dots\right) \\ &\quad + \left(1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \dots\right) \end{aligned}$$

begins with 1 and contains no terms involving powers of x up to (and including) x^6 .

6. The n th derivative of $f(x)$ is $f^{(n)}(x) = 2^n e^{2x}$ so $f^{(n)}(1) = 2^n e^2$ and the Taylor series is

$$\begin{aligned} \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} (x-1)^n &= \sum_{n=0}^\infty \frac{2^n e^2}{n!} (x-1)^n \\ &= e^2 + 2e^2(x-1) + 2e^2(x-1)^2 + \frac{4}{3}e^2(x-1)^3 + \dots \end{aligned}$$

7. This power series is obtained from the power series

$$\frac{1}{1-x} = \sum_{k=0}^\infty x^k$$

by differentiating twice. Hence it represents the function

$$\frac{d^2}{dx^2} \left(\frac{1}{1-x} \right) = \frac{d}{dx} \left(\frac{1}{(1-x)^2} \right) = \frac{2}{(1-x)^3}.$$