

## Math 424 Spring 2006

### Handout on homeomorphisms, bi-Lipschitz maps and isometries

When are two metric spaces “the same”? More precisely, what sort of equivalence relations can we define on the collection of all metric spaces? The natural answer to this question depends on what sort of properties one is interested in. In this handout, we describe three notions of equivalence between metric spaces.

- *Homeomorphism* (topologists’ viewpoint)
- *Bi-Lipschitz map* (analysts’ viewpoint)
- *Isometry* (geometers’ viewpoint)

Let  $(X, d)$  and  $(Y, d')$  be metric spaces, and let  $f : X \rightarrow Y$  be one-to-one and onto.

**Definition 1.**  $f$  is a *homeomorphism* if  $f$  and  $f^{-1}$  are both continuous.

**Definition 2.**  $f$  is a *bi-Lipschitz transformation* if  $f$  and  $f^{-1}$  are both Lipschitz.

In other words,  $f$  is bi-Lipschitz if there exists a number  $L < \infty$  so that

$$\frac{1}{L}d(x, y) \leq d'(f(x), f(y)) \leq Ld(x, y) \quad (3)$$

for all  $x, y \in X$ .

**Definition 4.**  $f$  is an *isometry* if

$$d'(f(x), f(y)) = d(x, y)$$

for all  $x, y \in X$ .

Note that this is just the case  $L = 1$  in (3). Thus every isometry is a bi-Lipschitz transformation. Since Lipschitz functions are continuous, every bi-Lipschitz transformation is a homeomorphism. In short

$$\text{isometry} \Rightarrow \text{bi-Lip} \Rightarrow \text{homeomorphism}$$

None of these arrows can be reversed. Later on in this handout we will give examples of bi-Lipschitz maps which are not isometries, and homeomorphisms which are not bi-Lipschitz.

Each of these notions gives rise to a corresponding equivalence relation on metric spaces. For example, two metric spaces  $(X, d)$  and  $(Y, d')$  are *isometric* if there is an isometry  $f : X \rightarrow Y$ . (Similarly, define notions of *bi-Lipschitz equivalent* and *homeomorphic*.)

Isometric metric spaces are basically identical—there is no way to distinguish two isometric metric spaces  $X$  and  $Y$  using only the metric. Indeed, the isometry defines a one-to-one pairing between the elements of  $X$  and the elements of  $Y$  which preserves the metric. Basically, we can think of an isometry just as a relabelling of the names of the elements of  $X$ , inducing no change in the metric.

Bi-Lipschitz equivalent metric spaces are not quite the same, but in many situations can be treated as “almost” the same. A bi-Lipschitz transformation between two metric spaces  $X$  and

$Y$  defines a one-to-one pairing between the elements of  $X$  and  $Y$  which only changes the metric by a universal bounded factor  $L$ .

The most flexible of these three notions is the concept of homeomorphism. These transformations can distort the metric tremendously, but still do so in a continuous manner.

We can distinguish the various properties of metric spaces which we introduced in Chapter III according to whether they are *topological* or *metric properties*. Topological properties are unchanged under arbitrary homeomorphisms; metric properties are unchanged under arbitrary bi-Lipschitz maps, but may change under more general homeomorphisms. (All properties of metric spaces are preserved by isometries.)

*Topological properties:* compactness, connectedness (proved in sections 4.4, 4.5, respectively).

*Metric properties:* boundedness, completeness. Exercise: show that these properties are preserved by bi-Lipschitz maps.

Neither boundedness nor completeness is a topological property. See Example 7.

**Example 5.**  $X = (0, 1)$  and  $Y = (2, 3)$  are isometric, via the map  $f : X \rightarrow Y$ ,  $f(x) = x + 2$ .

**Example 6.** Let  $X = (0, 1)$  and  $Y = (0, 100)$ . Then  $X$  and  $Y$  are not isometric, but are bi-Lipschitz via the map  $f : X \rightarrow Y$ ,  $f(x) = 100x$ .

**Example 7.** Let  $X = (0, 1)$  and  $Y = \mathbb{R}$ . Then  $X$  and  $Y$  are not bi-Lipschitz equivalent, but are homeomorphic via the map  $f : X \rightarrow Y$ ,

$$f(x) = \frac{x - \frac{1}{2}}{x(1-x)}.$$

Note that  $X$  is bounded but not complete, while  $Y$  is complete but not bounded.

Let  $X = \{0, 1\}^{\mathbb{N}}$  be the set of all countable sequences of 0's and 1's. Write elements of  $X$  in the form  $x = (x_1, x_2, \dots)$ , where each  $x_n \in \{0, 1\}$ . The function  $d : X \times X \rightarrow \mathbb{R}$  defined by  $d(x, y) = 0$  if  $x = y$ , and

$$d(x, y) = 3^{-\min\{k \in \mathbb{N} : x_k \neq y_k\}}$$

if  $x \neq y$ , where  $x = (x_1, x_2, \dots)$  and  $y = (y_1, y_2, \dots)$ , is a metric on  $X$ . (This was one of the problems on Exam I.) Let  $\mathbf{C} \subset \mathbb{R}$  be the Cantor set, with the Euclidean metric. Recall that this is the closed set

$$\mathbf{C} = [0, 1] \setminus \left( \left( \frac{1}{3}, \frac{2}{3} \right) \cup \left( \frac{1}{9}, \frac{2}{9} \right) \cup \left( \frac{7}{9}, \frac{8}{9} \right) \cup \left( \frac{1}{27}, \frac{2}{27} \right) \cup \left( \frac{7}{27}, \frac{8}{27} \right) \cup \dots \right).$$

**Theorem 8.**  $(X, d)$  and  $\mathbf{C}$  are bi-Lipschitz equivalent.

We will prove this in class. The idea, however, is remarkably simple. Indeed, we will show that the function

$$f(x_1, x_2, \dots) = \frac{2x_1}{3} + \frac{2x_2}{9} + \frac{2x_3}{27} + \dots$$

is a bi-Lipschitz transformation from  $(X, d)$  onto  $\mathbf{C}$ . This gives an abstract, symbolic description for the Cantor set.

**Corollary 9.** The Cantor set  $\mathbf{C}$  is uncountable.

*Proof.* Every bi-Lipschitz transformation is (in particular) one-to-one and onto. Thus bi-Lipschitz equivalent spaces have the same cardinality. In particular,  $\mathbf{C}$  has the same cardinality as  $\{0, 1\}^{\mathbb{N}}$ . But  $\{0, 1\}^{\mathbb{N}}$  has the same cardinality as the set  $P(\mathbb{N})$  of all subsets of  $\mathbb{N}$ , via the one-to-one and onto map  $x = (x_1, x_2, \dots) \mapsto \{i \in \mathbb{N} : x_i = 1\}$ . Thus  $\mathbf{C}$  has the same cardinality as  $P(\mathbb{N})$ . We proved earlier that  $P(\mathbb{N})$  is uncountable (cardinality exceeds that of  $\mathbb{N}$ ).  $\square$