Math 230 Section BE1/BB1
Lab 8: Improper Integrals

There are two kinds of *improper* integrals:

(i) Integrals over infinite intervals.

(ii) Integrals whose integrand has an infinite discontinuity within the interval of integration.

An integral may exhibit both of these types of improper behavior.

Examples 1. Type (i):

$$\int_1^\infty \frac{dx}{x^2}, \quad \int_{-\infty}^0 e^x\,dx, \quad \int_{-\infty}^\infty \frac{dx}{1+x^2}.$$

Type (ii):

$$\int_{-1}^1 \frac{dx}{x^4}, \quad \int_0^1 \frac{dx}{\sqrt{x}}, \quad \int_0^e \ln x\,dx.$$

Types (i) and (ii) together:

$$\int_0^\infty \frac{dx}{x^{1/3}}, \quad \int_{-\infty}^\infty \frac{x\,dx}{(x+1)(x-2)}.$$

The values of improper integrals are defined as limits of proper integrals. Such a limit need not exist.

First, we define improper integrals over infinite intervals.

Definition 2. Let f be a continuous function on the interval $[a,\infty)$. The *(improper) integral* of f over this interval is

$$\int_a^\infty f(x)\,dx = \lim_{c\to\infty}\int_a^c f(x)\,dx.$$

The integral *converges* if the limit exists and is finite, and *diverges* if the limit does not exist or is infinite. Similarly, the *(improper) integral* of a continuous function f over $(-\infty,b]$ is

$$\int_{-\infty}^b f(x)\,dx = \lim_{c\to\infty}\int_c^b f(x)\,dx$$

and the *(improper) integral* of f over $(-\infty, \infty)$ is

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx.$$

Next, we define improper integrals whose integral has an infinite discontinuity.

Definition 3. Let f be continuous on an interval $[a, b)$ with an infinite discontinuity at $x = b$. The *(improper) integral* of f over the interval $[a, b]$ is

$$\int_{a}^{b} f(x)\,dx = \lim_{c \to b^-} \int_{a}^{c} f(x)\,dx.$$

The integral *converges* if the limit exists and is finite, and *diverges* if the limit does not exist or is infinite.

Similarly, if f is continuous on $(a, b]$ with an infinite discontinuity at $x = a$, the *(improper) integral* of f over $[a, b]$ is

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{c \to \infty} \int_{c}^{b} f(x)\,dx.$$

Finally, if f is continuous on $[a, c) \cup (c, b]$ with an infinite discontinuity at $x = c$, then the *(improper) integral* of f over $[a, b]$ is

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx.$$

The following examples are extremely important. You should make sure that you are very familiar with the convergence/divergence of integrals of powers of x over intervals with endpoints at 0 and/or infinity.

Examples 4.

$$\int_{1}^{\infty} \frac{dx}{x^p} = \lim_{c \to \infty} \int_{1}^{c} \frac{dx}{x^p} = \lim_{c \to \infty} \left. \frac{x^{1-p}}{1-p} \right|_{1}^{c} = \frac{1}{p-1} \lim_{c \to \infty} \left(1 - \frac{1}{c^{p-1}}\right) \tag{1}$$

which exists and equals $1/(p-1)$, if $p > 1$ and does not exist if $p \leq 1$.

$$\int_{0}^{1} \frac{dx}{x^p} = \lim_{c \to 0^+} \int_{c}^{1} \frac{dx}{x^p} = \lim_{c \to 0^+} \left. \frac{x^{1-p}}{1-p} \right|_{c}^{1} = \frac{1}{1-p} \lim_{c \to 0^+} \left(1 - c^{1-p}\right) \tag{2}$$

which exists and equals $1/(1-p)$, if $p < 1$ and does not exist if $p \geq 1$.

To summarize:

$$\int_1^\infty \frac{dx}{x^p} \quad \begin{array}{c}\text{converges if } p > 1 \\ \text{diverges if } p \le 1\end{array}$$

and

$$\int_0^1 \frac{dx}{x^p} \quad \begin{array}{c}\text{converges if } p < 1 \\ \text{diverges if } p \ge 1\end{array}.$$

The Laplace transform. A *transform* is an operation which converts one function into another. (Differentiation is an obvious example.)

The *Laplace transform* is an important transform which arises in the theory of differential equations. The Laplace transform of a function $f(t)$ is a new function, written $\mathcal{L}f(s)$, which is defined by an improper integral:

$$\mathcal{L}f(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

The domain of $\mathcal{L}f(s)$ consists of all values of s for which this improper integral converges.

Example 5. The Laplace transform of the constant function $f(t) = 1$ is the function $\mathcal{L}f(s) = 1/s$, with domain $s > 0$. To see this, compute

$$\mathcal{L}f(s) = \int_0^\infty e^{-st}(1)\, dt = \lim_{c\to\infty} \int_0^c e^{-st}\, dt = \lim_{c\to\infty} \left.\frac{e^{-st}}{-s}\right|_{t=0}^{t=c} = \frac{1}{s}(1 - \lim_{c\to\infty} e^{-sc}) = \frac{1}{s}$$

provided $s > 0$.

Remark 6. Finally, let's discuss the *comparison tests* for improper integrals. Suppose that $f(x)$ and $g(x)$ are continuous functions defined on an interval $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $a \le x$. Then:

1. If $\int_a^\infty f(x)$ diverges, then $\int_a^\infty g(x)$ diverges.
2. If $\int_a^\infty g(x)$ converges, then $\int_a^\infty f(x)$ converges.

These facts are obvious from the interpretation of the integral as the area underneath the curve. There are similar comparison tests for improper integrals on intervals of the form $(-\infty, b]$, $(-\infty, \infty)$, improper integrals with infinite discontinuities, etc.

Exercises

Exercise 1. For each integral, tell if it converges or diverges. If it converges, give the value of the integral.

(a) $\int_0^\infty \frac{dx}{1+x^2}$

(b) $\int_0^\infty \frac{x\,dx}{1+x^2}$

(c) $\int_0^{\pi/2} \sec^2 x\,dx$

(d) $\int_0^\infty e^{-x}\,dx$

(e) $\int_0^\infty xe^{-x}\,dx$

(f) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Exercise 2. For which values of p does the improper integral $\int_0^\infty e^{px}\,dx$ converge? For which values of p does this integral diverge?

Exercise 3. Find the area of the region bounded above by the graph of the function $y = 6/(x^2 - 9)$, below by the line $y = 0$, and on the left by the line $x = 5$.

Exercise 4. Consider the (infinite) region bounded above by the graph of the function $y = 1/x$, below by the line $y = 0$ and on the left by the line $x = 1$. Let S be the solid of revolution obtained by revolving this region about the x-axis.

(a) Write an integral to compute the volume of S. Show that this integral converges, and compute the volume of S.

(b) Write an integral to compute the area of the surface which bounds S. Show that this integral diverges.

Exercise 5. For each of the following functions $f(t)$, compute the Laplace transform $\mathcal{L}f(s)$, and give the domain of $\mathcal{L}f(s)$.

(a) $f(t) = e^{-t}$.

(b) $f(t) = t$.

(c) (Extra credit) $f(t) = \sin t$.