Mastery Exam I: Solutions

1. *(20 points) Let* $f(x) = \ln(1+x)$ *and let* $I = \int_1^5 f(x)\,dx$.

   (a) *Explain why* $|f'(x)| \le \frac{1}{2}$ *and* $|f''(x)| \le \frac{1}{4}$ *for all* x *in* $[1,5]$.

   $|f'(x)| = \frac{1}{1+x}$ is a decreasing function on $[1,5]$ so

   $$|f'(x)| \le |f'(1)| = \frac{1}{1+1} = \frac{1}{2}$$

   $|f''(x)| = \left|-\frac{1}{(1+x)^2}\right| = \frac{1}{(1+x)^2}$ is also a decreasing function on $[1,5]$ so

   $$|f''(x)| \le |f''(1)| = \frac{1}{(1+1)^2} = \frac{1}{4}$$

   (b) *Use part (a) to find error bounds for* R_{10} *and* M_5.

   Since $|f'(x)| \le \frac{1}{2}$ we may choose $K_1 = \frac{1}{2}$. Then

   $$\left|\int_1^5 f(x)\,dx - R_{10}\right| \le \frac{K_1(b-a)^2}{2n} = \frac{\frac{1}{2}(5-1)^2}{2(10)} = \frac{2}{5}$$

   Since $|f''(x)| \le \frac{1}{4}$ we may choose $K_2 = \frac{1}{4}$. Then

   $$\left|\int_1^5 f(x)\,dx - M_5\right| \le \frac{K_2(b-a)^3}{24n^2} = \frac{\frac{1}{4}(5-1)^3}{24(5^2)} = \frac{2}{75}$$

2. *(15 points) Consider the IVP* $y' = x^2 + y$ *with* $y(0) = 1$.

   (a) *Use Euler's method with one step of size 4 to estimate* $y(4)$.

x	y	y'
0	1	$0^2 + 1 = 1$
4	$1 + (1)(4) = 5$	

   $$y(4) \approx 5$$

   (b) *Use Euler's method with four steps of size 1 to estimate* $y(4)$.

x	y	y'
0	1	$0^2 + 1 = 1$
1	$1 + (1) = 2$	$1^2 + 2 = 3$
2	$2 + (3)(1) = 5$	$2^2 + 5 = 9$
3	$5 + (9)(1) = 14$	$3^2 + 14 = 23$
4	$14 + (23)(1) = 37$	

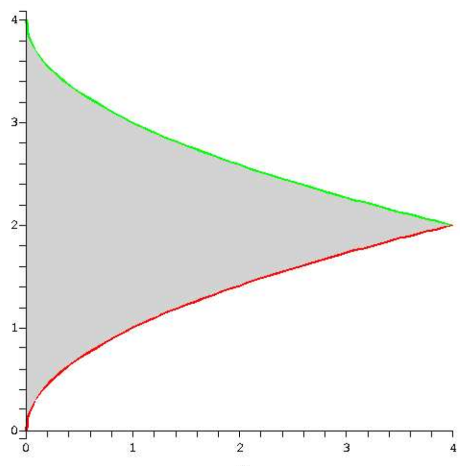
   $$y(4) \approx 37$$

3. *(15 points) Find the arc length of the graph of the function* $y = 2x^{3/2}$ *between* $x = 0$ *and* $x = 11$.

The arc length is given by the integral

$$\begin{aligned}\int_0^{11} \sqrt{1 + (3\sqrt{x})^2}\, dx &= \int_0^{11} \sqrt{1+9x}\, dx \\ &= \int_0^{11} (1+9x)^{1/2}\, dx \\ &= \frac{1}{9}\, \frac{1+9x}{3/2}\Big|_0^{11} \\ &= \frac{2}{27}\left(100^{3/2} - 1^{3/2}\right) = 74\end{aligned}$$

4. *(15 points) Sketch the region bounded by the graphs of* $y = \sqrt{x}$, $y = 4 - \sqrt{x}$, $x = 0$ *and* $x = 4$. *Find the volume of the solid obtained by revolving this region about the* x*-axis.*



$$\begin{aligned}V = \pi\int_0^4 (4-\sqrt{x})^2 - (\sqrt{x})^2\, dx &= \pi\int_0^4 16 - 8\sqrt{x} + x - x\, dx \\ &= \pi\left[16x - \frac{16}{3}x^{3/2}\right]_0^4 \\ &= \pi\left(64 - \frac{128}{3}\right) = \frac{64}{3}\pi\end{aligned}$$

5. *Find the solution to the IVP* $y' = (1+x)(1+y)$, $y(0) = 1$. *Give your answer in the form* $y = f(x)$.

$$\begin{aligned}
\frac{dy}{dx} &= (1+x)(1+y) \\
\int \frac{dy}{1+y} &= \int (1+x)\,dx \\
\ln|1+y| &= x + \frac{x^2}{2} + C \\
|1+y| &= e^{x+\frac{x^2}{2}+C} \\
1+y &= Ae^{x+x^2/2} \\
y &= -1 + Ae^{x+x^2/2}
\end{aligned}$$

Since $y(0) = 1$ we get $A = 2$ so

$$y = -1 + 2e^{x+x^2/2}$$

6. (20 points)

(a) Find $\int x^2 e^{5x}\,dx$. Check your answer by differentiation.

$$\begin{aligned}
\int x^2 e^{5x}\,dx &= \frac{x^2e^{5x}}{5} - \frac{2}{5}\int xe^{5x}\,dx \\
&= \frac{x^2e^{5x}}{5} - \frac{2}{5}\left[\frac{xe^{5x}}{5} - \frac{1}{5}\int e^{5x}\,dx\right] \\
&= e^{5x}\left(\frac{x^2}{5} - \frac{2x}{25} + \frac{2}{125}\right) + C
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{dx}e^{5x}\left(\frac{x^2}{5} - \frac{2x}{25} + \frac{2}{125}\right) + C \\
&\quad = 5e^{5x}\left(\frac{x^2}{5} - \frac{2x}{25} + \frac{2}{125}\right) \\
&\quad\quad + e^{5x}\left(\frac{2x}{5} - \frac{2}{25}\right) \\
&\quad = x^2e^{5x}
\end{aligned}$$

(b) Find $\int x^3 \ln x \, dx$. Check your answer by differentiation.

$$\begin{aligned}\int x^3 \ln x \, dx &= \frac{x^4 \ln x}{4} - \int \frac{x^3}{4} \, dx \\ &= \frac{x^4 \ln x}{4} - \frac{x^4}{16} + C\end{aligned}$$

$$\frac{d}{dx}\left(\frac{x^4 \ln x}{4} - \frac{x^4}{16} + C\right) = x^3 \ln x + \frac{x^3}{4} - \frac{x^3}{4} = x^3 \ln x$$