Math 490
In class exercise on polyhedra

Let P be a polyhedron (not necessarily regular) in $\mathbf{R}^3$, with V vertices, E edges and F faces. For each $i = 3, 4, 5, \ldots$, let F_i be the number of i-gon faces (i.e., F_3 is the number of triangles, F_4 is the number of squares, etc.), and for each $j = 3, 4, 5, \ldots$, let V_j be the number of vertices at which j faces meet (i.e., V_3 is the number of trivalent vertices, etc.). Thus

$$F = \sum_{i=3}^{\infty} F_i \qquad \text{and} \qquad V = \sum_{j=3}^{\infty} V_j.$$

(1) Explain why

$$2E = \sum_{i=3}^{\infty} iF_i = \sum_{j=3}^{\infty} jV_j.$$

(2) Prove that

$$2\sum_{i=3}^{\infty} F_i - \sum_{j=3}^{\infty}(j-2)V_j = 4 \qquad \text{and} \qquad 2\sum_{j=3}^{\infty} V_j - \sum_{i=3}^{\infty}(i-2)F_i = 4.$$

(3) Use part (2) to show that

$$\sum_{i=3}^{\infty}(4-i)F_i + \sum_{j=3}^{\infty}(4-j)V_j = 8.$$

Conclude that every polyhedron has either triangular faces or trivalent vertices. For each choice of a pair of positive even integers a and b with $a + b = 8$, find a polyhedron with

$$F_3 = a, F_5 = F_6 = \cdots = 0$$

and

$$V_3 = b, V_5 = V_6 = \cdots = 0.$$

(4) Using part (2) again, show that

$$\sum_{i=3}^{\infty}(6-i)F_i - \sum_{j=3}^{\infty}(2j-6)V_j = 12.$$

Conclude that

(i) every polyhedron has either triangles, squares or pentagons,
(ii) if a polyhedron has no triangles or squares, it has at least 12 pentagons,
(iii) if a polyhedron has no triangles or pentagons, it has at least 6 squares,
(iv) if a polyhedron has no squares or pentagons, it has at least 4 triangles.

Show that the results in parts (ii), (iii) and (iv) are sharp, i.e., find a polyhedron with only the given number of polygons of the appropriate type.