

Math 230 Spring 2005

Lab 2: Simpson's Rule and Quadratic Approximation

The goal of this lab is to investigate Simpson's Rule for approximating integrals, and understand how it arises via *quadratic approximation*.

To begin, let's recall the **Midpoint** and **Trapezoid Rules**. For any positive integer n , the integral

$$\int_a^b f(x) dx$$

can be approximated by either of the following sums:

$$M_n := \sum_{i=1}^n f\left(\frac{x_{i-1} + x_i}{2}\right) \Delta x_i = \sum_{i=1}^n f(m_i) \Delta x_i$$

or

$$T_n := \sum_{i=1}^n \frac{f(x_{i-1}) + f(x_i)}{2} \Delta x_i.$$

Here

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

is a *partition* of the interval $[a, b]$ into (possibly unequally sized) subintervals, $\Delta x_i = x_i - x_{i-1}$ is the width of the i th subinterval, and $m_i = (x_{i-1} + x_i)/2$ is the midpoint of the i th subinterval.

We've discussed these approximations in detail already. See Figure 2 on page 379 of the text for an illustration of the Midpoint and Trapezoid approximations of $\int_0^1 x^2 dx$.

Simpson's Rule approximates the integral $\int_a^b f(x) dx$ by the quantity

$$S_{2n} := \frac{2}{3}M_n + \frac{1}{3}T_n.$$

As discussed in the textbook (page 380), we can think of this as a *weighted average* of the Midpoint and Trapezoid approximations. We give M_n twice as much weight as T_n because, typically, the error committed by T_n is about twice as much as that committed by M_n .

Exercise 1. Consider the integral $I = \int_0^\pi \sin(x) dx$.

(a) Find the exact value of I .

(b) Calculate the Midpoint and Trapezoid approximations to I with two equal subdivisions of the interval $[0, \pi]$, i.e., compute M_2 and T_2 .

(c) Calculate Simpson's approximation S_4 . What is the (absolute) error committed by S_4 ?

Now return to the general formulas for M_n and T_n on page 1. Assume the subdivisions are of equal size: $\Delta x_i = \Delta x = (b - a)/n$.

Exercise 2. (a) Show that

$$S_{2n} = \Delta x \left(\frac{f(x_0)}{6} + \frac{2f(m_1)}{3} + \frac{f(x_1)}{3} + \cdots + \frac{f(x_{n-1})}{3} + \frac{2f(m_n)}{3} + \frac{f(x_n)}{6} \right).$$

Hint: it may help to write out the first few and last few terms of M_n and T_n .

(b) Rename the $2n$ subdivision points

$$x'_0 = x_0, x'_1 = m_1, x'_2 = x_1, \dots, x'_{2n-2} = x_{n-1}, x'_{2n-1} = m_n, x'_{2n} = x_n$$

and let

$$\Delta x' = \frac{b - a}{2n}$$

be the width of the corresponding subintervals. Rewrite the above expression for S_{2n} in terms of these expressions.

Fitting Curves with Quadratic Functions

You already know how to fit a line through two points. Given two points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$, the function

$$y = L(x) = y_0 + m(x - x_0), \quad m = \frac{y_1 - y_0}{x_1 - x_0},$$

is the unique linear function whose graph passes through P_0 and P_1 .

If $h = \Delta x = x_1 - x_0$ is the difference between the x -values of P_0 and P_1 , then

$$\int_{x_0}^{x_1} L(x) dx = y_0 h + \frac{1}{2} m h^2 = h(y_0 + \frac{1}{2} m h) = h(y_0 + \frac{1}{2}(y_1 - y_0)) = \frac{1}{2} h(y_0 + y_1).$$

Notice that the result $\frac{1}{2}h(y_0 + y_1)$ is just the value of the Trapezoid approximation to $\int_{x_0}^{x_1} L(x) dx$ with the trivial partition $x_0 < x_1$. In other words, the Trapezoid approximation is exactly correct (commits no error) when applied to linear functions.

An analogous procedure can be used to fit a quadratic function through three points. Given three points $P_0 = (x_0, y_0)$, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ there is a unique quadratic function

$$y = Q(x) = y_0 + m(x - x_0) + \frac{1}{2}\sigma(x - x_0)^2$$

whose graph passes through P_0 , P_1 and P_2 . To simplify matters, let's assume that the x -coordinates of these points are equally spaced: $h = \Delta x = x_1 - x_0 = x_2 - x_1$. Then the values of m and σ in the expression for $Q(x)$ are

$$m = -\frac{y_2 - 4y_1 + 3y_0}{2h} \quad \text{and} \quad \sigma = \frac{y_2 - 2y_1 + y_0}{h^2}.$$

Rather than working this out in general, consider the following special case.

Exercise 3. Fit a quadratic function

$$y = Q(x) = y_0 + m(x - x_0) + \frac{1}{2}\sigma(x - x_0)^2$$

through the points $P_0 = (x_0, y_0) = (1, 10)$, $P_1 = (x_1, y_1) = (2, 15)$ and $P_2 = (x_2, y_2) = (3, 24)$. Show that the above expressions for m and σ are correct in this case. Then compute

$$\int_1^3 Q(x) dx$$

and show that it is equal to

$$\frac{1}{3}h(y_0 + 4y_1 + y_2).$$

Extra Credit (a lot of algebra!): Show that the above expressions for m and σ are correct in general, and prove that the identity

$$\int_{x_0}^{x_2} Q(x) dx = \frac{1}{3}h(y_0 + 4y_1 + y_2)$$

always holds true.

Exercise 4. Show how the expression for Simpson's approximation which you found in Exercise 2 arises by adding up the expressions

$$\frac{1}{3}h(f(x'_{2i-2}) + 4f(x'_{2i-1}) + f(x'_{2i}))$$

for $i = 1, 2, \dots, n$. Explain what this represents geometrically. How are we approximating the area under the graph of $y = f(x)$ from $x = a$ to $x = b$? (Hint: take a look at page 403 in the textbook, but write your answer in your own words.)

Exercise 5. (a) Verify that Simpson's Rule computes the integral of every quadratic function exactly by completing the following table. In the first column, give the exact value of the integral. In the second column, the values of Simpson's approximation S_2 are given, using the partition $x'_0 = a$, $x'_1 = (a + b)/2$, and $x'_2 = b$ with $h = (b - a)/2$. In each line, simplify the expression in the second column and show that it is equal to the value in the first column.

Integral	Exact value	Simpson's approximation S_2
$\int_a^b 1 dx$		$\frac{1}{3}h(1 + 4 + 1)$
$\int_a^b x dx$		$\frac{1}{3}h(a + 4\frac{a+b}{2} + b)$
$\int_a^b x^2 dx$		$\frac{1}{3}h(a^2 + 4(\frac{a+b}{2})^2 + b^2)$

Exercise 5 shouldn't come as a surprise; your answer to Exercise 4 showed that Simpson's approximation works by replacing the graph of $y = f(x)$ on an interval $[x'_{2i-2}, x'_{2i}]$ by a suitable quadratic approximation. So if $y = f(x)$ is already a quadratic function, Simpson's approximation is in fact an exact formula.

In fact, Simpson's Rule also computes the integrals of *cubic* functions exactly. This is a somewhat unexpected bonus; it's not at all clear from the above discussion why it should be true.

Exercise 5. (b) Verify that Simpson's Rule computes the integral of every cubic function exactly by adding the following line to the table:

Integral	Exact value	Simpson's approximation S_2
$\int_a^b x^3 dx$		$\frac{1}{3}h(a^3 + 4(\frac{a+b}{2})^3 + b^3)$

Next time, we'll discuss error bounds for all five approximation schemes (left and right Riemann sums, midpoint and trapezoid approximations, and Simpson's Rule) in detail. As a preview, finish with the following problem.

Exercise 6. Go back to the integral $I = \int_0^\pi \sin(x) dx$ which you considered in Exercise 1.

(a) Calculate the Midpoint and Trapezoid approximations to I with four equal subdivisions of the interval $[0, \pi]$, i.e., compute M_4 and T_4 . Then calculate Simpson's approximation S_8 and the absolute error committed by S_8 .

(b) Compare your answer with the answer you got to Exercise 1(c). You doubled the number of subdivisions (from $n = 2$ to $n = 4$). How much smaller is the error committed by S_8 than the error committed by S_4 ? Can you make a conjecture about the effect of doubling the number of subdivisions on the error?