Problem 1 (30 points)

A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win $1.5 = \frac{3}{2}$ dollars. If they are different colors, you win -1 dollar. (That is you lose 1 dollar.) Calculate the expected value and the variance of the amount you win.

Let X the amount you win. Then X takes only two values -1 and $\frac{3}{2}$.

$$P\{X=\tfrac{3}{2}\} = P\{\text{the two marbles have the same color}\}$$

$$= P\{\text{2 red balls are withdrawn}\} + P\{\text{2 blue marbles are withdrawn}\}$$

$$= \frac{\binom{5}{2}}{\binom{10}{2}} + \frac{\binom{5}{2}}{\binom{10}{2}} = 2\frac{\binom{5}{2}}{\binom{10}{2}} = 2\frac{\frac{5!}{\cancel{2!}3!}}{\frac{10!}{8!\cancel{2!}}} = \cancel{2}\frac{\cancel{5}\cdot 4}{9\cdot 1\cancel{0}} = \frac{4}{9}$$

$$P\{X=-1\} = 1 - P\{X=\tfrac{3}{2}\} = \frac{5}{9} = \frac{\binom{5}{1}\binom{5}{1}}{\binom{10}{2}}$$

$$E(X) = \frac{3}{2}P\{X=\tfrac{3}{2}\} - 1\,P\{X=-1\} = \frac{3}{2}\cdot\frac{4}{9} - \frac{5}{9} = \frac{2}{3} - \frac{5}{9} = \frac{6}{9} - \frac{5}{9} = \frac{1}{9}$$

$$[E(X)]^2 = \frac{1}{81}$$

$$E(X^2) = \left(\frac{3}{2}\right)^2 P\{X=\tfrac{3}{2}\} + (-1)^2 P\{X=-1\} = \frac{9}{4}\frac{4}{9} + \frac{5}{9} = \frac{14}{9}$$

$$\mathrm{Var}(X) = \frac{14}{9} - \frac{1}{9^2} = \frac{1}{9}\left(14 - \frac{1}{9}\right) = \frac{125}{81}.$$

Problem 2 (25 points)

The density function of the random variable X is given by

$$f(x) = \begin{cases} a + bx^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E[X] = \frac{3}{5}$, find a and b.

Sine $f(x)$ is a density: $\int_{-\infty}^{\infty} f(x)\,dx = 1$ or

$1 = \int_0^1 (a+bx^2)\,dx = ax + \frac{bx^3}{3}\Big|_0^1 = a + \frac{b}{3}$ Thus $a + \frac{b}{3} = 1$ (1)

$\frac{3}{5} = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x(a+bx^2)\,dx = \int_0^1 (ax + bx^3)\,dx = \frac{ax^2}{2} + \frac{bx^4}{4}\Big|_0^1$

$= \frac{a}{2} + \frac{b}{4} \Rightarrow a + \frac{b}{2} = \frac{6}{5}$

$a + \frac{b}{3} = 1$

$\Rightarrow \frac{b}{6} = \frac{1}{5} \Rightarrow \boxed{b = \frac{6}{5}}$

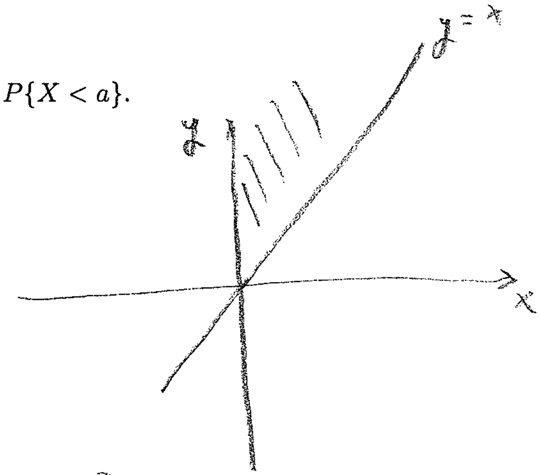
$a = \frac{6}{5} - \frac{b}{2} = \frac{6}{5} - \frac{3}{5} \Rightarrow \boxed{a = \frac{3}{5}}$

Problem 3 (25 points)

The joint probability density function of X and Y is given by

$$f(x, y) = e^{-(x+y)}$$

for $0 \le x < \infty$, $0 \le y < \infty$. Find $P\{X < Y\}$ and $P\{X < a\}$.

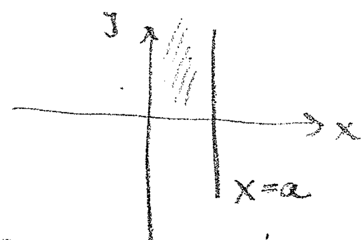


a) $P\{X<Y\} = \iint\limits_{x<y} f(x,y)\,dx\,dy$

$$= \int_0^\infty \int_0^y e^{-x} e^{-y}\, dx\, dy$$

$$= \int_0^\infty e^{-y}\left(\int_0^y e^{-x}\, dx\right) dy = \int_0^\infty e^{-y}\left[-e^{-x}\right]_0^y dy = \int_0^\infty (1-e^{-y})\,e^{-y}\, dy$$

$$= \int_0^\infty e^{-y} dy - \int_0^\infty e^{-2y} dy = -e^{-y}\Big|_0^\infty + \frac{1}{2} e^{-2y}\Big|_0^\infty = 1 - \frac{1}{2} = \frac{1}{2}$$



b) $P\{X<a\} = \iint\limits_{x<a} f(x,y)\,dx\,dy$

$$= \int_0^a \int_0^\infty e^{-x} e^{-y}\, dx\, da = \left(\int_0^a e^{-x}\, dx\right)\left(\int_0^\infty e^{-y}\, dy\right) = 1 - e^{-a}$$

Problem 4 (20 points)

a) (10 points) The probability mass function of a binomial random variable with parameters (n, p) is given by

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 0, 1, 2, ..., n.$$

Find the expected value of the binomial random variable with parameters (n, p).

Hint: You may use the identity $i\binom{n}{i} = n\binom{n-1}{i-1}$ and the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

b) (10 points). The probability mass function of a Poisson random variable with parameter λ is given by

$$p(i) = e^{-\lambda}\frac{\lambda^i}{i!}, \quad i = 0, 1, 2, ...$$

Show that the Poisson random variable can be used as an approximation for a binomial random variable with parameters (n, p) when n is large and $np = \lambda$ is of moderate size.

a) $E(X) = \sum_{i=0}^{n} i\,p(i) = \sum_{i=0}^{n} i\binom{n}{i} p^i (1-p)^{n-i} = \sum_{i=1}^{n} i\binom{n}{i} p^i (1-p)^{n-i}$

$= \sum_{i=1}^{n} n\binom{n-1}{i-1} p^i (1-p)^{n-i} = n\sum_{i=1}^{n} \binom{n-1}{i-1} p\,p^{i-1} (1-p)^{n-1-(i-1)}$

$= np\sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} = np\sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k}$

$= np\,(p + 1 - p)^{n-1} = np$

b) $p(i) = P(X=i) = \binom{n}{i} p^i (1-p)^{n-i} = \frac{n!}{i!(n-i)!} \frac{(np)^i}{n^i} \left(1 - \frac{\lambda}{n}\right)^{n-i}$

$= \frac{\lambda^i}{i!}\left(1 - \frac{\lambda}{n}\right)^n \frac{n!}{(n-i)!\,n^i\left(1 - \frac{\lambda}{n}\right)^i}$

But $\frac{n!}{(n-i)!} = (n-i+1)(n-i+2)\ldots n \approx n^i$ for n large

and thus $\frac{n!}{(n-i)!\, n^i} \approx 1$ for n large, $\left(1-\frac{\lambda}{n}\right)^i \approx 1$

Since $\left(1-\frac{\lambda}{n}\right)^n \approx e^{-\lambda}$ we have that $p(i) \approx \frac{\lambda^i}{i!} e^{-\lambda}$ for n large