Homework #10 Section 7.1 # 2,5, Section 7.2 # 2,3
Section 7.3 #2

7.1 #2 $\Delta u = 0$ in $D \subset \mathbb{R}^3$

$\frac{\partial u}{\partial n} = g$ on ∂D (1)

Set $v = u_1 - u_2$ where u_1, u_2 are solutions of (1)

Then $\Delta v = 0$ in D.

$$\frac{\partial v}{\partial n} = \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = g - g = 0 \quad \text{on } \partial D$$

By Green's Identity $\iint_{\partial D} v \frac{\partial v}{\partial n} \, dS = \iiint_D |\nabla v|^2 \, dX = 0$

since $\frac{\partial v}{\partial n} = 0$ on ∂D. Thus $\nabla v = 0$ or $v = C$.

Thus $u_1 = C + u_2$ or $u_1 - u_2 = C$ and we have uniqueness up to a constant.

#5 Let $E(w) = \frac{1}{2} \iiint_D |\nabla w|^2 dx - \iint_{\partial D} h w \, dS$

Let $\Delta u = 0$ in D.

$\frac{\partial u}{\partial n} = h$ on ∂D

We want to prove that among all real-valued functions $w(x)$ on D the quantity $E(w) = \frac{1}{2}\iiint_D |\nabla w|^2 dx - \iint_{\partial D} hw\, dS$ is the smallest for $w = u$, where u solves

$$\Delta u = 0 \text{ in } D$$
$$\frac{\partial u}{\partial n} = h \quad \text{on } \partial D$$

Let $v = u - w \Rightarrow w = u - v$. Then

$$E(w) = \frac{1}{2}\iiint_D |\nabla(u-v)|^2 dx - \iint_{\partial D} h(u-v)\, dS = \frac{1}{2}\iiint_D |\nabla u|^2 dx + \frac{1}{2}\iiint_D |\nabla v|^2 dx$$

$$- \iiint_D (\nabla u \cdot \nabla v)\, dx + \iint_{\partial D} hv\, dS - \iint_{\partial D} hu\, dS$$

$$= E(u) + \frac{1}{2}\iiint_D |\nabla v|^2 dx - \iiint_D (\nabla u \cdot \nabla v)\, dx + \iint_{\partial D} hv\, dS$$

By Green's identity

$$\iiint_D (\nabla u \cdot \nabla v)\, dx = \iint_{\partial D} v \frac{\partial u}{\partial n}\, ds - \iiint_D v \overbrace{\Delta u}^{0}\, dx = \iint_{\partial D} vh\, ds$$

$$\Rightarrow \quad 0 = -\iiint_D (\nabla u \cdot \nabla v)\, dx + \iint_{\partial D} vh\, ds$$

Thus $E(w) = E(u) + \frac{1}{2}\iiint_D |\nabla v|^2 dx \geq E(u)$ and it is smallest for $w = u$ in which case $E(w) = E(u)$.

Section 7.2

#2 Let $\varphi(x)$ be any C^2 function defined on $\mathbb{R}^3$ but vanishes outside some ball. Show that

$$\varphi(0) = -\iiint_{\mathbb{R}^3} \frac{1}{|x|} \Delta \varphi(x) \frac{dx}{4\pi}$$

or

$$\iiint_{\mathbb{R}^3} -\Delta\left(\frac{1}{|x|}\right) \varphi(x)\, dx = 4\pi\,(\varphi, \delta) \quad \text{or}$$

$$\left(-\Delta\left(\frac{1}{|x|}\right), \varphi\right) = (4\pi\delta, \varphi) \quad \text{or} \quad -\Delta\left(\frac{1}{|x|}\right) = 4\pi\,\delta(x)$$ in the distributional sense.

Proof: Recall second Green's Identity

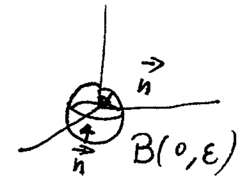
$$\iiint_D (u\Delta v - v\Delta u)\, dx = \iint_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS$$

We know that $-\Delta\left(\frac{1}{|x|}\right) = 0$ for $x \neq 0$. Thus we apply the identity where $D_\varepsilon = \mathbb{R}^3 - B(0,\varepsilon)$, $u = \varphi$, $v = -\frac{1}{|x|}$ where $B(0,\varepsilon)$ is the ball in 3d, centered at zero with radius ε.

Since $\Delta v = \Delta\left(-\frac{1}{|x|}\right) = 0$ in D_ε we have

$$\iiint_{D_\varepsilon} \frac{1}{|x|} \Delta \varphi(x)\, dx = \iint_{\partial D_\varepsilon} \varphi \frac{\partial}{\partial n}\left(-\frac{1}{|x|}\right) dS + \iint_{\partial D_\varepsilon} \frac{1}{|x|} \frac{\partial \varphi}{\partial n}\, dS$$

The boundary of D_ε is the boundary of the ball $B(0,\varepsilon)$ which is the sphere $\partial B(0,\varepsilon)$.



$\vec{n} = -\dfrac{\vec{x}}{|\vec{x}|}$ and on $\partial D_\varepsilon := \partial B(0,\varepsilon)$, $|x| = \varepsilon$.

Thus $$\iiint_{\mathbb{R}^3 - B(0,\varepsilon)} \frac{1}{|x|}\,\Delta\varphi(x)\,dx = \frac{1}{\varepsilon}\iint_{\partial B(0,\varepsilon)} \frac{\partial \varphi}{\partial n}\,dS + \iint_{\partial B(0,\varepsilon)} \varphi\,\frac{\partial}{\partial n}\left(-\frac{1}{|x|}\right) dS \quad (1)$$

Now $$\frac{1}{\varepsilon}\iint_{\partial B(0,\varepsilon)} \frac{\partial\varphi}{\partial n}\,dS \leq \frac{1}{\varepsilon}\iint_{\partial B(0,\varepsilon)} |n\cdot\nabla\varphi|\,dS \leq \frac{1}{\varepsilon}\iint_{\partial B(0,\varepsilon)} |\nabla\varphi|\,dS$$

$$\leq \max_{x\in\partial B(0,\varepsilon)} |\nabla\varphi|\;\frac{1}{\varepsilon}\iint_{\partial B(0,\varepsilon)} dS = \left(\max_{x\in\partial B(0,\varepsilon)} |\nabla\varphi|\right)\frac{4\pi\varepsilon^2}{\varepsilon} = 4\pi\varepsilon\left(\max|\nabla\varphi|\right)$$

As $\varepsilon\to 0$ we have $\displaystyle\frac{1}{\varepsilon}\iint_{\partial B(0,\varepsilon)} \frac{\partial\varphi}{\partial n}\,dS \to 0$.

But $$\frac{\partial}{\partial n}\left(-\frac{1}{|x|}\right) = n\cdot\nabla\left(-\frac{1}{|x|}\right) = -\frac{x}{|x|}\cdot\nabla\left(-\frac{1}{|x|}\right) = \frac{1}{|x|^2} = \frac{1}{\varepsilon^2}$$

and thus $$\lim_{\varepsilon\to 0}\iint_{\partial B(0,\varepsilon)} \varphi\,\frac{\partial}{\partial n}\left(-\frac{1}{|x|}\right) dS = 4\pi \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon^2}\iint_{\partial B(0,\varepsilon)} \varphi\,dS$$

$$= 4\pi\lim_{\varepsilon\to 0}\frac{1}{|\partial B(0,\varepsilon)|}\iint_{\partial B(0,\varepsilon)}\varphi\,dS = 4\pi\varphi(0)$$ since φ is continuous (actually it is C^2)

Thus taking $\varepsilon \to 0$ in (1) we have

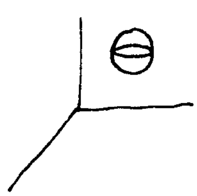
$$\iiint_{\mathbb{R}^3} \frac{1}{|x|} \Delta\varphi(x)\,dx = \lim_{\varepsilon \to 0} \iiint_{\mathbb{R}^3 - B(0,\varepsilon)} \frac{1}{|x|} \Delta\varphi(x)\,dx = 4\pi\,\varphi(0)$$ and we are

done.

#3 We know that the solution of $\Delta u = 0$ in D is given by

$$u(x_0) = \iint_{\partial D} \left[u(x) \frac{\partial}{\partial n}\left(-\frac{1}{|x-x_0|}\right) + \frac{1}{|x-x_0|} \frac{\partial u}{\partial n} \right] \frac{dS}{4\pi}$$

Now let $D = B(x_0, a)$, the ball of radius a, centered at x_0



On $\partial B(x_0, a)$ we have $\vec{n} = \dfrac{x - x_0}{|x-x_0|}$ and

$$\frac{\partial}{\partial n}\left(-\frac{1}{|x-x_0|}\right) = \vec{n} \cdot \nabla\left(-\frac{1}{|x-x_0|}\right) = \frac{1}{|x-x_0|^2} = \frac{1}{a^2}$$

Thus $$u(x_0) = \iint_{\partial B(x_0,a)} u(x) \frac{1}{4\pi a^2}\,dS + \iint_{\partial B(x_0,a)} \frac{1}{4\pi a} \frac{\partial u}{\partial n}\,dS$$

$$= \frac{1}{4\pi a^2} \iint_{\partial B(x_0,a)} u(x)\,dS + \frac{1}{4\pi a} \iint_{\partial B(x_0,a)} \frac{\partial u}{\partial n}\,dS$$

We know that $\iint_{\partial D} v \frac{\partial u}{\partial n}\, dS = \iiint_D \nabla v \cdot \nabla u \, dx + \iiint_D v \Delta u \, dx$

Let $v=1$ then $\iint_{\partial D} \frac{\partial u}{\partial n}\, dS = \iiint_D \Delta u \, dx$. But for $D = B(x_0, a)$, u

is harmonic thus $\Delta u = 0$. Thus $\iint_{\partial B(x_0,a)} \frac{\partial u}{\partial n}\, dS = 0$ and

$u(x_0) = \frac{1}{|\partial B(x_0,a)|} \iint_{\partial B(x_0,a)} u(x)\, dS$ which is the mean value theorem.

Section 7.3 #2 Solve $\begin{array}{l} \Delta u = f \text{ in } D \\ u = h \text{ on } \partial D \end{array}$ using the Green's function.

Exactly the way we derived the representation formula for the problem $\begin{array}{l} \Delta u = 0 \text{ in } D \\ u = h \text{ on } \partial D \end{array}$ in class and found that

$u(x_0) = \iint_{\partial D} \left[-h(x) \frac{\partial}{\partial n}\left(\frac{1}{|x-x_0|} \right) + \frac{1}{|x-x_0|} \frac{\partial u}{\partial n} \right] \frac{dS}{4\pi}$ we can prove that

the representation formula for the problem $\begin{array}{l} \Delta u = f \text{ in } D \\ u = h \text{ on } \partial D \end{array}$ is

given by $u(x_0) = \iint_{\partial D} \left[-h(x) \frac{\partial}{\partial n}\left(\frac{1}{|x-x_0|} \right) + \frac{1}{|x-x_0|} \frac{\partial u}{\partial n} \right] \frac{dS}{4\pi} - \frac{1}{4\pi} \iiint_D \frac{1}{|x-x_0|} f \, dx$ (1)

Now let $v(x) = -\frac{1}{4\pi |x-x_0|}$ and $G(x, x_0) = v(x) + H(x)$

where $G(x, x_0)$ is the Green's function. (This is the definition of H)

We know that H is harmonic in all of D and thus applying the second Green's identity $\iiint_D u\Delta v - v\Delta u = \iint_{\partial D} \left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right] dS$ with $v = H$ we have

$$\iiint_D -H\Delta u = -\iiint_D Hf\,dx = \iint_{\partial D} \left[u\frac{\partial H}{\partial n} - H\frac{\partial u}{\partial n}\right] dS \quad (2)$$

or

$$0 = \iiint_D Hf\,dx + \iint_{\partial D} \left[h(x)\frac{\partial H}{\partial n} - H\frac{\partial u}{\partial n}\right] dS \quad (3).$$

By the representation formula we have

$$u(x_0) = \iint_{\partial D} \left[h(x)\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right] dS + \iiint_D v(x) f(x)\,dX \quad (4)$$

Adding (3) + (4) we obtain

$$\underline{u(x_0)} = \iiint_D [H(x) + v(x)] f(x)\,dX + \iint_{\partial D} \left(u\frac{\partial G}{\partial n} - \frac{\partial u}{\partial n} G\right) dS$$

$$= \iiint_D G(x, x_0) f(x)\,dX + \iint_{\partial D} u\frac{\partial G}{\partial n}\,dS \quad \text{since } G = 0 \text{ on } \partial D.$$