MATH 461 : HW #10

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Problem 7.45

a.) Let $Y = X_1 + X_2$, $Z = X_2 + X_3$; then $Var(Y) = Var(X_1) + Var(X_2) = 1+1 = 2$
$Var(Z) = Var(X_2) + Var(X_3) = 1+1 = 2$

$Cov(Y,Z) = Cov(X_1+X_2, X_2+X_3) = Cov(X_1, X_2) + Cov(X_1, X_3) + Cov(X_2, X_3) + Var(X_2) = 1$

$$\therefore \rho = \frac{Cov(Y,Z)}{\sqrt{Var(Y)\cdot Var(Z)}} = 1/2$$

b.) Let $Y = X_1 + X_2$, $Z = X_3 + X_4$, Then $Var(Y) = 2$, $Var(Z) = 2$; $Cov(Y,Z) = 0$
$\because Cov(X_i, X_j) = 0$ for $i \neq j$

$\therefore \rho = 0$

Problem 7.48 a.) X is a Geometric Random Variable with probability of success $p = 1/6$
$\therefore E\{X\} = 1/p = 6$

b.) $E\{X \mid Y=1\} = 1 + E\{X\}$ because the outcomes after the 1st roll (which is a 5) can be considered equivalent to the original sequence $\therefore E\{X \mid Y=1\} = 1+6 = 7$

c.) $\{X \mid Y=5\} = \{X \text{ s.t. } X<5 \mid Y=5\} \cup \{X \text{ s.t } X>5 \mid Y=5\}$

$$\therefore E\{X \mid Y=5\} = \sum_{x=1}^{4} (1-p)^{x-1}\cdot p \cdot x + \underbrace{(1-p)^4\cdot(5+6)}$$

here $p = \text{Prob.}(\text{a "6"} \mid \text{no "5" occurs})$
$= \frac{1/6}{5/6} = 1/5$

once no 5 or 6 are obtained on first 4 rolls and a 5 is obtained on 5th roll, the remaining sequence of die-rolls can be considered equivalent to the original sequence without conditioning

$$\therefore E\{X \mid Y=5\} = 1\cdot\frac{1}{5} + 2\cdot\frac{4}{5}\cdot\frac{1}{5} + 3\cdot(4/5)^2\cdot 1/5 + 4\cdot(4/5)^3\cdot 1/5 + (4/5)^4\cdot(5+6) = 5.8192$$

Problem 7.50 $$f_{X|Y}(x|y) = \frac{e^{-x/y}\cdot e^{-y}/y}{\int_0^\infty e^{-x/y}\cdot e^{-y}/y \cdot dx} = \frac{1}{y}\cdot e^{-x/y}$$
$0 < x < \infty$

$\therefore X \mid Y=y$ is Exponential Random Variable with mean y

$$\therefore E\{X^2 \mid Y=y\} = Var(X \mid Y=y) + (E\{X \mid Y=y\})^2 = y^2 + y^2 = 2y^2$$

Problem 7.51 $$f_{X|Y}(x|y) = \frac{e^{-y}/y}{\int_0^y e^{-y}/y \cdot dx} = \frac{1}{y}$$; $0 < x < y$ $\therefore X \mid Y=y$ is Uniform $(0,y)$

$$E\{X^3 \mid Y=y\} = \int_0^y x^3 \frac{1}{y} dx = y^3/4$$

Problem 7·75 $M_X(t) = e^{2(e^t-1)} \Rightarrow X \sim \text{Poisson}(\lambda=2)$ $\quad p_X(k) = \frac{e^{-2}\cdot 2^k}{k!}$; $k \geq 0$

$M_Y(t) = \left(\frac{1}{4}+\frac{3}{4}e^t\right)^{10} \Rightarrow Y \sim \text{Binomial}(n=10, p=3/4)$ $\quad p_Y(k) = \binom{10}{k}(3/4)^k(1/4)^{10-k}$; $0 \leq k \leq 10$

a.) $P(X+Y=2) = P(X=0, Y=2) + P(X=1, Y=1) + P(X=2, Y=0)$

$= e^{-2}\binom{10}{2}(.75)^2(.25)^8 + 2\cdot e^{-2}\binom{10}{1}(.75)(.25)^9 + 2\cdot e^{-2}(.25)^{10}$

$= e^{-2}(.25)^{10}\left[405 + 60 + 2\right] \approx 6.0274 \times 10^{-5}$

b.) $P(XY=0) = P(\{X=0\} \cup \{Y=0\}) = P(X=0) + P(Y=0) - P(X=0, Y=0)$

$= e^{-2} + (.25)^{10} - e^{-2}(.25)^{10} \approx 0.13534$

c.) $E\{X\cdot Y\} = E\{X\}\cdot E\{Y\} = \lambda \cdot np = 2 \times 10 \times \frac{3}{4} = 15$

Theoretical Problem 7·23 $Z \sim N(0,1)$ $\therefore E\{Z\}=0,\ E\{Z^2\} = 1 = Var(Z)$

$Var(Z^2) = E\{Z^4\} - 1$; $E\{Z^4\} = M^{(4)}(t)\big|_0$ where $M(t) = e^{t^2/2}$

$\therefore Var(Z^2) = 2$ $\qquad = (3+6t^2+t^4)e^{t^2/2}\big|_0 = 3$

$Cov(Z^2, Z) = E\{(Z^2-1)(Z-0)\} = E\{Z^3\} - E\{Z\} = E\{Z^3\} = M^{(3)}(t)\big|_0$

$= (3t+t^3)e^{t^2/2}\big|_0 = 0$

$\rho(Y,Z) = \frac{Cov(a+bZ+cZ^2, Z)}{\sqrt{Var(a+bZ+cZ^2)\cdot Var(Z)}} = \frac{b\cdot Var(Z) + c\cdot Cov(Z^2, Z)}{\sqrt{[b^2 Var(Z) + c^2 Var(Z^2) + 2bc\cdot Cov(Z^2,Z)]\cdot 1}}$

$= \frac{b(1) + c(0)}{\sqrt{b^2(1) + c^2(2) + 2bc(0)}} = \frac{b}{\sqrt{b^2+2c^2}}$ Hence proved.

Problem 8·1 $P(0 < X < 40) = P(-20 < X-20 < 20) = 1 - P(|X-20| \geq 20)$

$\geq 1 - \frac{20}{20^2} = 19/20$ $\quad \therefore \boxed{P(0<X<40) \geq 19/20}$

by Chebyshev's Inequality

Problem 8·2

a.) $P(X \geq 85) \leq \frac{E\{X\}}{85} = \frac{75}{85} = \frac{15}{17}$ by Markov's Inequality

b.) $P(65 \leq X \leq 85) = P(-10 \leq X-75 \leq 10) = 1 - P(|X-75| > 10)$

$\geq 1 - \frac{25}{10^2} = 3/4$ by Chebyshev's Inequality

c.) Let n = # students, $\bar{X} = \frac{\sum X_i}{n}$; $E\{\bar{X}\} = 75$, $Var(\bar{X}) = 25/n$

$P(|\bar{X}-75| \leq 5) \geq 0.9 \Rightarrow 1 - P(|\bar{X}-75| > 5) \geq 0.9 \Rightarrow P(|\bar{X}-75| > 5) \leq 0.1$

by Chebyshev's Inequality $P(|\bar{X}-75| > 5) \leq \frac{25/n}{5^2} = \frac{1}{n}$ $\therefore$ we must have $\frac{1}{n} \leq 0.1$ or $\boxed{n \geq 10}$

Problem 8.4 $\quad \sum_{i=1}^{20} X_i \sim \text{Poisson}(\lambda = 20 \times 1) \quad \therefore E\{\sum X_i\} = 20$

$\text{Var}(\sum X_i) = 20$

a.) $P(\sum X_i > 15) \leqslant \frac{E\{\sum X_i\}}{15}$ by Markov's Inequality

$= \frac{20}{15} = \frac{4}{3}$

b.) by the Central Limit Theorem

$P(\sum X_i > 15) \approx P(Z > \frac{15-20}{\sqrt{20}}) = P(Z > -1.118)$ where $Z \sim N(0,1)$

$\simeq 1 - \Phi_Z(-1.12) = 0.8686$

Problem 8.5 For $i = 1, \ldots, 50$

let E_i be the error in the approximation of the ith number

Then $\{E_i\}_{i=1}^{50}$ are IID and Uniform $(-1/2, 1/2)$ $\therefore E\{E_i\} = 0, \text{Var}(E_i) = 1/12$

$\Rightarrow E\{\sum_{i=1}^{50} E_i\} = 0, \quad \text{Var}(\sum_{i=1}^{50} E_i) = 50/12 = 25/6$

$\therefore P(|\sum E_i| > 3) \approx P(|N(0,1)| > \frac{3-0}{\sqrt{\frac{25}{6}}})$ by Central Limit Theorem

$= 2 \cdot P(N(0,1) > 3\sqrt{6}/5) = 2 \cdot (1 - \Phi(3\sqrt{6}/5)) = 0.1416$