HW #2 Section 1.5 # 1, 4a), b)
Section 1.6 # 6
Section 2.1 # 7, 8

MATH 442

#1 $\frac{d^2u}{dx^2} + u = 0$ We know the general solution of the ODE is $u(x) = C_1 \cos x + C_2 \sin x$

$0 < x < L$

$u(0) = 0$
$u(L) = 0$

Since $u(0) = 0 \Rightarrow C_1 = 0$

Thus $u(x) = C \sin x$.

Moreover $0 = u(L) = C \sin L$.

If $\sin L \neq 0 \Leftrightarrow L \neq n\pi$, $n \in \mathbb{Z}$ (thus if $L$ is not an integer multiple of $\pi$)

then $C = 0$ and $u(x) = 0$ is the only solution.

But if $L = n\pi$ then $u(x) = \sin x$ is another solution and we don't have uniqueness.

#4 a) $\Delta u = f(x,y,z)$ in $D$

$\frac{\partial u}{\partial n} = 0$ on $\partial D$

Let $u$ be a solution. Then if $v = u + C$ where $C$ is an arbitrary constant, then $v$ is another solution since

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#4 $\Delta u = f(x,y,z)$ in $D$

a) $\frac{\partial u}{\partial n} = 0$ on $\partial D$

Let $u$ be a solution. Then if $v = u + C$ where $C$ is an arbitrary constant, then $v$ is another solution since

$\Delta v = \Delta u = f$ in $D$

and $\frac{\partial v}{\partial n} = \frac{\partial u}{\partial n} + 0 = 0$ on $\partial D$

No uniqueness

b) Assume $u$ solves $\Delta u = f$ with $\frac{\partial u}{\partial n} = 0$ on $\partial D$.

Then $\iiint_D \Delta u \, dx\,dy\,dz = \iiint_D f \, dx\,dy\,dz$

By Gauss Theorem $\iiint_D \Delta u \, dx\,dy\,dz = \iiint_D \operatorname{div}(\vec{\nabla} u) \, dx\,dy\,dz$

$= \iint_{\partial D} (\vec{\nabla} u \cdot \vec{n}) \, dS = \iint_{\partial D} \cancel{\frac{\partial u}{\partial n}}^{\,0} \, dS = 0$.

Thus a necessary condition for the Neumann problem to have a solution is $\iiint_D f(x,y,z) \, dx\,dy\,dz = 0$.

#6 a) $3u_y + u_{xy} = 0$

We have $a_{11} = a_{22} = 0$ and $a_{12} = \frac{1}{2}$

Since $a_{12}^2 > a_{11}a_{22} = 0$ the PDE is hyperbolic.

b) Let $v = u_y$ then $3v + v_x = 0$ or

$v_x = -3v$. Thus $v(x,y) = f(y)\, e^{-3x}$

Now $\frac{\partial u}{\partial y}(x,y) = f(y)\, e^{-3x} \Rightarrow u(x,y) = e^{-3x} \int_0^y f(t)\,dt + g(x)$

where $f$ and $g$ are arbitrary functions.

c) Let $u(x,0) = e^{-3x}$. Then

$e^{-3x} = g(x)$ and thus $u(x,y) = e^{-3x}\left[1 + \int_0^y f(t)\,dt\right]$

If in addition $\frac{\partial u}{\partial y}(x,0) = 0$ we have

$\frac{\partial u}{\partial y}(x,y) = f(y)\, e^{-3x} \Rightarrow 0 = \frac{\partial u}{\partial y}(x,0) = f(0)\, e^{-3x}$

$\Rightarrow f(0) = 0$.

The solution is not unique. Any function of the form

$u(x,y) = e^{-3x}\left[1 + \int_0^y f(t)\,dt\right]$ with $f(0) = 0$ is a solution

For example, let $f(y) = y$ then $u(x,y) = e^{-3x}\left(1 + \frac{1}{2}y^2\right)$ is a solution, but if we take $f(y) = y^2$ then

$u(x,y) = e^{-3x}\left(1 + \frac{1}{3}y^3\right)$ is also a solution.

#7 The general solution is written

$$u(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

Assume $\varphi, \psi$ are odd functions. Then

$$u(-x,t) = \frac{1}{2}\left[\varphi(-x+ct) + \varphi(-x-ct)\right] + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)\,ds$$

$$= \frac{1}{2}\left[-\varphi(x-ct) - \varphi(x+ct)\right] + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)\,ds$$

If we make the change of variables $s' = -s$ for the last integral we obtain

$$u(-x,t) = -\frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] - \frac{1}{2c}\int_{x+ct}^{x-ct} \psi(-s')\,ds'$$

$$= -\left\{\frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] - \frac{1}{2c}\int_{x+ct}^{x-ct} \psi(s')\,ds'\right\}$$

$$= -\left\{\frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds\right\} = -u(x,t)$$

Thus for every $t$, $u(\cdot,t)$ is also odd.

#8 Consider $u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_{r}\right) = c^2 \Delta u$ (radial Laplacian in 3D)

a) Let $v = ru$. Then $v_r = ru_r + u$, $v_{rr} = ru_{rr} + 2u_r$ (*)

and $v_{tt} = ru_{tt}$.

Thus $v_{tt} = r u_{tt} = r c^2\left( u_{rr} + \frac{2}{r} u_r \right) = c^2\left( r u_{rr} + 2u_r \right) \overset{(*)}{=}$

$c^2 v_{rr}$.

Thus if $u$ is a solution to $u_{tt} = c^2\left( u_{rr} + \frac{2}{r} u_r \right)$ and we set $v = ru$ then $v$ satisfies the 1D Wave Equation.

b) The general solution to the 1D wave equation is $f(x+ct) + g(x-ct)$ where $f$ and $g$ are arbitrary functions.

Thus $v(r,t) = f(r+ct) + g(r-ct)$ and thus

$u(r,t) = \frac{1}{r}\left[ f(r+ct) + g(r-ct) \right]$. Notice that $v(0,t) = 0$

c) If $u(r,0) = \varphi(r)$ and $u_t(r,0) = \phi(r)$ then

$\varphi(r) = \frac{1}{r}\left( f(r) + g(r) \right) \to \quad f(r) + g(r) = r\varphi(r)$

and $\phi(r) = u_t(r,0) = \frac{c}{r}\left[ f'(r) - g'(r) \right]$

Thus $f'(r) + g'(r) = (r\varphi(r))'$ (1)

$f'(r) - g'(r) = \frac{r\phi(r)}{c}$ (2)

Thus $f'(r) = \frac{1}{2} \frac{r\phi(r)}{c} + \frac{1}{2}(r\varphi(r))'$

$g'(r) = \frac{1}{2} \frac{r\phi(r)}{c} - \frac{1}{2}(r\varphi(r))'$

We integrate exactly the same way as when we solved the 1D wave equation in class and obtain with the help of $u(r,t) = \frac{1}{r}\left[f(r+ct) + g(r-ct)\right]$ that

$$u(r,t) = \frac{1}{2r}\left\{(r+ct)\varphi(r+ct) + (r-ct)\varphi(r-ct)\right\} + \frac{1}{2cr}\int_{r-ct}^{r+ct} s\psi(s)\,ds$$

Notice that $v(r,t) = (r+ct)\varphi(r+ct) + (r-ct)\varphi(r-ct) + \frac{1}{2cr}\int_{r-ct}^{r+ct} s\psi(s)\,ds$

and if $\varphi$ and $\psi$ are even functions then

$$v(0,t) = ct\,\varphi(ct) - ct\,\varphi(-ct) + \frac{1}{2cr}\int_{-ct}^{ct} s\psi(s)\,ds$$

$$= 0$$ since $\varphi$ is even and $s\psi(s)$ is odd.