HW #3 Section 2.2 #1, #2
Section 2.3 #4
Section 2.4 #8 and #15

#1 Consider
$$u_{tt} = c^2 u_{xx}$$
$$u(x,0) = \varphi(x)$$
$$u_t(x,0) = \psi(x)$$

Let $E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\,dx$. without loss of generality $\rho = T = c = 1$.

Then
$$\frac{dE}{dt} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{d}{dt}\left(u_t^2 + u_x^2\right)dx = \int_{-\infty}^{\infty}(u_t u_{tt} + u_x u_{tx})\,dx$$

$$\overset{u_{tt}=u_{xx}}{=} \int_{-\infty}^{\infty} (u_t u_{xx} + u_x u_{tx})\,dx = \int_{-\infty}^{\infty} u_t u_{xx}\,dx - \int_{-\infty}^{\infty} u_t u_{xx} + \left[u_t u_x\right]_{-\infty}^{\infty}$$

by integration by parts. If we assume that at $\pm\infty$ $u = 0$ then

$u_t u_x \Big|_{-\infty}^{\infty} = 0$ and thus $\frac{dE}{dt} = 0$ or $E(u(t)) = E(u(0))$

Note that if we assume φ and ψ to be compactly supported by finite speed of propagation, u is zero at $\pm\infty$.

Now take $\varphi = \psi = 0$. Then $E(u(0)) = 0$ and thus

$$\frac{1}{2}\int_{-\infty}^{\infty} u_t^2\,dx + \frac{1}{2}\int_{-\infty}^{\infty} u_x^2\,dx = 0$$

Thus $u_t(x,t) = 0$ or $u(t,x) = f(x)$. But

$$0 = \varphi(x) = u(0,x) = f(x) \quad \Rightarrow \quad u(t,x) = 0.$$

#2 a) Let $e = \frac{1}{2}(u_t^2 + u_x^2)$, $P = u_t u_x$

We have $\frac{\partial e}{\partial t} = u_t u + u_x u_{xt} \overset{u_{tt}=u_{xx}}{=} u_t u_{xx} + u_x u_{xt}$

$$= \frac{\partial}{\partial x}(u_t u_x) = \frac{\partial P}{\partial x}$$

On the other hand

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial t}(u_t u_x) = u_{tt} u_x + u_t u_{xt} \overset{u_{tt}=u_{xx}}{=}$$

$$u_x u_{xx} + u_t u_{xt} = \frac{\partial}{\partial x}\left(\frac{1}{2}u_x^2 + \frac{1}{2}u_t^2\right) = \frac{\partial e}{\partial x}$$

b) Since $e_t = P_x$ and $e_x = P_t$ we have

$$\left.\begin{array}{l} e_{tt} = P_{tx} \\ e_{xx} = P_{xt} \end{array}\right. \Rightarrow e_{tt} = e_{xx}$$ by the equality of the mixed derivatives.

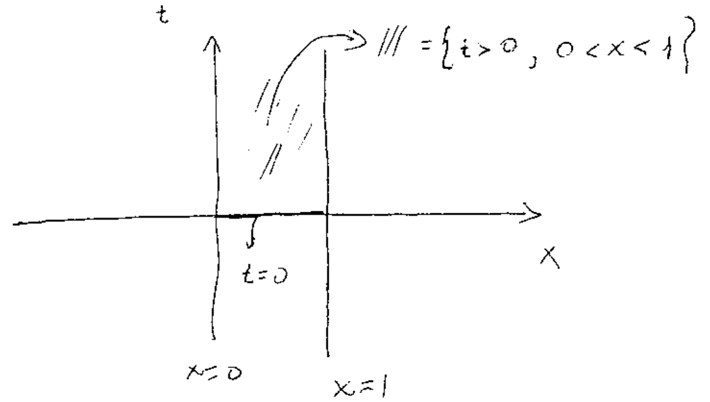
Similarly $$\left.\begin{array}{l} P_{tt} = e_{xt} \\ P_{xx} = e_{tx} \end{array}\right\} \Rightarrow P_{tt} = P_{xx}$$ and both e and P satisfy the wave equation.

(#4) Strong-maximum principle: If $u(x,t)$ satisfies the diffusion equation in a rectangle $(0 \le x \le l,\ 0 \le t \le T)$ then the maximum cannot be assumed anywhere inside the rectangle but only on $t=0$ or $x=0$ or $x=l$ unless u is constant.

a) $u_t = u_{xx}$, $0 < x < 1$, $0 < t < \infty$

$u(x,0) = 4x(1-x)$

$u(0,t) = u(1,t) = 0$



Notice that at $t=0$ we have $u(x,0) = 4x(1-x)$

$$\frac{du}{dx}(x,0) = 4 - 8x = 0 \iff x = \frac{1}{2}$$

$$\frac{d^2u}{dx^2}(x,0) = -8 < 0$$. Thus $u(\frac{1}{2},0) = 4 \cdot \frac{1}{2}(1-\frac{1}{2}) = 1$ is a local maximum. Since $0<x<1$ we know that 1 is the absolute maximum. Thus the maximum value of $u(t,x)$ on the boundary is 1 and the minimum is zero. (This is obvious since $u(0,t) = u(1,t) = 0$)

Thus $0 \le u(t,x) \le 1$. But in $t>0$, $0<x<1$ (inside the shaded region) $u(t,x)$ cannot be 1 or 0 because this would contradict the strong maximum principle and the minimum principle. Thus $0 < u(t,x) < 1$ for $t>0$, $0<x<1$.

b) Now let $t \geq 0$, $0 \leq x \leq 1$

If u satisfies $u_t = u_{xx}$ then $v(t,x) = u(1-x,t)$ satisfies the same equation.

Thus we consider $w(x,t) = u(x,t) - u(1-x,t)$

$$w(x,0) = u(x,0) - u(1-x,0) = 4x(1-x) - 4(1-x)(1-(1-x))$$
$$= 4x(1-x) - 4x(1-x) = 0$$

$$w(0,t) = u(0,t) - u(1,t) = 0$$
$$w(1,t) = u(1,t) - u(0,t) = 0$$

Thus $W_t = W_{xx}$ with $w(0,t) = w(1,t) = 0$
$w(x,0) = 0$

By the maximum principle $w(t,x) \geq 0$. By the minimum principle $w(t,x) \leq 0$. Thus $w(t,x) = 0$

or $u(t,x) = u(t,1-x)$.

c) Let $E(t) = \int_0^1 u^2(t,x)\,dx$

$$\frac{dE}{dt} = \int_0^1 2uu_t\,dx = 2\int_0^1 u u_{xx}\,dx = 2[uu_x]_0^1$$
$$- 2\int_0^1 u_x^2\,dx$$ Since $u(0,t) = u(1,t) = 0$, $[uu_x]_0^1 = 0$

Thus $\frac{dE}{dt} = -2\int_0^1 u_x^2\,dx \leqslant 0$

But actually $\frac{dE}{dt} < 0$ since if $\int_0^1 u_x^2\,dx = 0 \Rightarrow$

$u_x(x,t) = 0 \Rightarrow u(t,x) = g(t)$ and $u(t,0) = 0 = g(t)$.

Thus $u(x,t)$ is identically zero. (which is not true)

Thus $\frac{dE(t)}{dt} < 0$ and $\int_0^1 u^2(t,x)\,dx$ is strictly decreasing.

#8 Let $S(t,x) = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/4kt} \geq 0$

Fix any $\delta > 0$. Then for $|x| \geq \delta$

$$S(t,x) \leqslant \frac{1}{\sqrt{4\pi k t}} e^{-\delta^2/4kt}$$

. As $\underset{t>0}{t \to 0}$ we know that $\frac{1}{t} \to +\infty$. But $\boxed{\lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{4\pi k}} e^{-\frac{\delta^2}{4k}x} = 0}$

Thus $\lim_{\substack{t\to 0\\ t>0}} \frac{1}{\sqrt{4\pi k t}} e^{-\delta^2/4kt} = 0$ and thus

$$\lim_{t\to 0} \max_{\delta \leq |x| < \infty} S(t,x) = 0,$$

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$$u_t - k u_{xx} = f(x,t)$$
$$u_x(0,t) = g(t)$$
$$u_x(l,t) = h(t)$$
$$u(x,0) = \varphi(x)$$

Consider two solution u and v. Then $w = u - v$ satisfies

$$w_t = k w_{xx}$$
$$w(x,0) = 0 \quad , \quad w(0,t) = w(l,t) = 0$$

Consider $E(t) = \frac{1}{2}\int_0^l w^2\,dx$.

$$\frac{dE}{dt} = \int_0^l w w_t\,dx = \int_0^l k w w_{xx}\,dx = \Big[k w w_x\Big]_0^l - k\int_0^l w_x^2\,dx$$

But since $w(0,t) = w(l,t) = 0$, $\Big[k w w_x\Big]_0^l = 0$ and

$$\frac{dE}{dt} = -k\int_0^l w_x^2\,dx \le 0 \quad \Rightarrow \quad E\big(w(t)\big) \le E\big(w(x,0)\big) = 0$$

Thus $\int_0^l w^2\,dx = 0 \quad \Rightarrow \quad w = 0 \quad \Rightarrow \quad u(x,t) = v(x,t)$.