HW 4

Section 3.1 #2, 3

Section 3.2 #1

Section 3.3 #1

#2 $u_t = k u_{xx}$

$u(x,0) = 0$

$u(0,t) = 1 \quad , 0 < x < \infty$

Let $v(x,t) = u(x,t) - 1$. If u solves $u_t = k u_{xx}$

Then $v_t = u_t$, $v_{xx} = u_{xx}$ and $v_t = k v_{xx}$

$\underline{v(x,0)} = u(x,0) - 1 = 0 - 1 = \underline{-1}$

$\underline{v(0,t)} = u(0,t) - 1 = \underline{0}$

By the method of reflection we solve using the formula (6) in the book with $\varphi(y) = -1$.

Then $$v(x,t) = \frac{-1}{\sqrt{4\pi k t}} \int_0^\infty \left[e^{-(x+y)^2/4kt} - e^{-(x-y)^2/4kt} \right] dy$$

$$= \frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-(x+y)^2/4kt}\, dy - \frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-(x-y)^2/4kt}\, dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4kt}}^{\infty} e^{-q^2}\, dq - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4kt}} e^{-p^2}\, dp = \frac{1}{2} - \frac{1}{2} \mathcal{E}rf\left(\frac{x}{\sqrt{4kt}}\right) - \left[\frac{1}{2} + \mathcal{E}rf\left(\frac{x}{\sqrt{4kt}}\right)\right]$$

Thus $v(x,t) = -\mathcal{E}rf\left(\frac{x}{\sqrt{4kt}}\right)$ and

$$u(x,t) = v(x,t) + 1 = 1 - \mathcal{E}rf\left(\frac{x}{\sqrt{4kt}}\right)$$

#3 Solve

$$W_t - k W_{xx} = 0$$
$$W_x(0,t) = 0 \qquad 0 < x < \infty$$
$$w(x,0) = \varphi(x)$$

Define $\varphi_{even}(x) = \begin{cases} \varphi(x) & x \geq 0 \\ \varphi(-x) & x \leq 0 \end{cases}$ and solve $\begin{array}{l} u_t = k u_{xx} \qquad -\infty < x < \infty \\ u(x,0) = \varphi_{even} \end{array}$

We know that $u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \varphi_{even}(y)\,dy$ and that

$u(x,t)$ is even since $\varphi_{even}(y)$ is even (Prove it as an exercise)

Thus $u_x(0,t) = 0$.

Now set $w(x,t) = u(x,t)$ for $x > 0$. Then

$$w(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \varphi_{even}(y)\,dy$$

$$= \frac{1}{\sqrt{4k\pi t}} \int_0^{\infty} \varphi(y)\, e^{-(x-y)^2/4kt}\,dy + \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{0} e^{-(x-y)^2/4kt} \varphi(-y)\,dy$$

$$\overset{y \mapsto -y}{=} \frac{1}{\sqrt{4k\pi t}} \int_0^\infty \left[e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt} \right] \varphi(y)\,dy$$

is the solution to the half-line Neumann Problem.

#1 Solve $w_{tt} - c^2 w_{xx} = 0 \qquad 0 < x < \infty$

$$w(x,0) = \varphi(x), \quad w_t(x,0) = \psi(x)$$

$$w_x(0,t) = 0$$

Define $\varphi_{even}(x) = \begin{cases} \varphi(x) & x \geq 0 \\ \varphi(-x) & x \leq 0 \end{cases}$

$\psi_{even}(x) = \begin{cases} \psi(x) & x \geq 0 \\ \psi(-x) & x \leq 0 \end{cases}$

Solve $u_{tt} - c^2 u_{xx} = 0$

$u(x,0) = \varphi_{even}(x) \qquad -\infty < x < \infty$ Then set $w(x,t) = u(x,t)$

$u_t(x,0) = \psi_{even}(x)$ for $x > 0$.

Thus $w(x,t) = \frac{1}{2}\left[\varphi_{even}(x+ct) + \varphi_{even}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{even}(y)\,dy$

<u>Case 1</u> Let $x > c|t|$. Then $x+ct > 0$ and $x-ct > 0$ and

$$w(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy$$

<u>Case 2</u> Let $0 < x < c|t|$. Two subcases

a) $t > 0$. Thus $x + ct > 0$, $x - ct < 0$

and

$$w(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(ct-x)\right] + \frac{1}{2c}\int_{x-ct}^{0} \psi_{even}(y)\,dy + \frac{1}{2c}\int_{0}^{x+ct} \psi_{even}(y)\,dy$$

$$\left(= \frac{1}{2c}\int_{x-ct}^{0} \psi(-y)\,dy + \frac{1}{2c}\int_{0}^{x+ct} \psi(y)\,dy \overset{y \mapsto -y}{=} \frac{1}{2c}\int_{0}^{ct-x} \psi(y)\,dy + \frac{1}{2c}\int_{0}^{x+ct} \psi(y)\,dy \right)$$

Thus $$w(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(ct-x)\right] + \frac{1}{2c}\int_{0}^{ct-x} \psi(y)\,dy + \frac{1}{2c}\int_{0}^{x+ct} \psi(y)\,dy$$

b) Let $t<0$. Then $x-ct>0$, $x+ct<0$

and $$w(x,t) = \frac{1}{2}\left[\varphi(x-ct) + \varphi(-x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{even}(y)\,dy$$

But $$\frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{even}(y)\,dy = -\frac{1}{2c}\int_{x+ct}^{x-ct} \psi_{even}(y)\,dy$$

$$= -\frac{1}{2c}\int_{x+ct}^{0} \psi_{even}(y)\,dy - \frac{1}{2c}\int_{0}^{x-ct} \psi_{even}(y)\,dy = -\frac{1}{2c}\int_{x+ct}^{0} \psi(-y)\,dy - \frac{1}{2c}\int_{0}^{x-ct} \psi(y)\,dy$$

$$\overset{y \mapsto -y}{=} -\frac{1}{2c}\int_{0}^{-x-ct} \psi(y)\,dy - \frac{1}{2c}\int_{0}^{x-ct} \psi(y)\,dy$$

Thus $$w(x,t) = \frac{1}{2}\left[\varphi(x-ct) + \varphi(-x-ct)\right] - \frac{1}{2c}\int_{0}^{-x-ct} \psi(y)\,dy - \frac{1}{2c}\int_{0}^{x-ct} \psi(y)\,dy$$

#1 Solve $u_t - k u_{xx} = f(x,t)$

$$u(x,0) = \varphi(x) \qquad 0 < x < \infty$$

$$u(0,t) = 0 \qquad 0 < t < \infty$$

The solution of $u_t - k u_{xx} = f(x,t)$, $-\infty < x < \infty$; $u(x,0) = \varphi(x)$, $0 < t$

is $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\varphi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s)\,dy\,ds$

Notice that if $\varphi(x)$ is odd and $f(x,t)$ is odd then $u(x,t)$ is odd.

Thus we define $\varphi_{odd}(x) = \begin{cases} \varphi(x), & x>0 \\ -\varphi(-x), & x<0 \\ 0, & x=0 \end{cases}$ $\quad f_{odd}(x,t) = \begin{cases} f(x,t), & x>0 \\ -f(-x,t), & x<0 \\ 0, & x=0 \end{cases}$

We then know that the solution to the original problem is for $x>0$

$$\underline{u(x,t)} = \int_{-\infty}^{\infty} S(x-y,t)\varphi_{odd}(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f_{odd}(y,s)\,dy\,ds$$

$$= \int_0^{\infty} S(x-y,t)\varphi(y)\,dy + \int_{-\infty}^{0} S(x-y,t)(-\varphi(-y))\,dy + \int_0^t \int_{-\infty}^{0} S(x-y,t-s)(-f(-y,s))\,dy\,ds$$

$$+ \int_0^t \int_0^{\infty} S(x-y,t-s) f(y,s)\,dy\,ds \overset{y \mapsto -y}{=}$$

$$\underline{\int_0^{\infty} [S(x-y,t) + S(x+y,t)]\varphi(y)\,dy + \int_0^t \int_0^{\infty} [S(x-y,t-s) - S(x+y,t-s)] f(y,s)\,dy\,ds}$$