HW5 Section 3.4 #1 Section 4.2 #1 and #2

Section 4.1 #4

#1 Solve $u_{tt} = c^2 u_{xx} + xt$, $u(x,0) = 0$, $u_t(x,0) = 0$

We know $u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} ys \, dy \, ds$

$$= \frac{1}{2c}\int_0^t s \left[\frac{y^2}{2} \right]_{x-c(t-s)}^{x+c(t-s)} ds = \frac{1}{2c}\int_0^t s A(s)\, ds \quad \text{where}$$

$$A(s) = \frac{(x+c(t-s))^2}{2} - \frac{[x-c(t-s)]^2}{2} = \frac{2x \cdot 2c(t-s)}{2} = 2cx(t-s)$$

Thus $u(x,t) = \frac{1}{\cancel{2c}}\int_0^t s \cancel{2c}\, x(t-s)\,ds = x\int_0^t (st - s^2)\,ds$

$$= x\left[\frac{s^2}{2}t - \frac{s^3}{3} \right]_0^t = \frac{xt^3}{6}$$

#4 $u_{tt} = c^2 u_{xx} - r u_t$, $0<x<l$

$u(0,t) = u(l,t) = 0$

$u(x,0) = \varphi(x)$, $u_t(x,0) = \psi(x)$ with $0 < r < \frac{\pi c}{l}$

$u(x,t) = X(x)T(t)$ $\quad u_{tt} = X(x)T''(t)$, $u_{xx} = X''(x)T(t)$ and

$X(x)T''(t) = c^2 X''(x)T(t) - r X(x) T'(t)$ or

$$\frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} - r\,\frac{T'(t)}{T(t)} \iff$$

$$-c^2 \frac{X''(x)}{X(x)} = -\frac{T''(t)}{T(t)} - r\,\frac{T'(t)}{T(t)} \iff -\frac{X''(x)}{X(x)} = -\frac{T''(t)}{c^2 T(t)} - \frac{r\,T'(t)}{c^2 T(t)} = \lambda$$

$$X''(x) + \lambda X(x) = 0$$

$X(0) = X(\ell) = 0$. We know that the solution is

$$X_n(x) = \sin\frac{n\pi x}{\ell}, \quad n = 1, 2, \dots \quad \text{and} \quad \lambda = \left(\frac{n\pi}{\ell}\right)^2$$

Now solve $\displaystyle -\frac{T''(t)}{c^2 T(t)} - r\,\frac{T'(t)}{c^2 T(t)} = \lambda = \left(\frac{n\pi}{\ell}\right)^2 \iff$

$$T''(t) + r\,T'(t) + \lambda c^2 T(t) = 0 \quad \text{characteristic polynomial}$$

$$\rho^2 + r\rho + \lambda c^2 = 0 \quad \text{or} \quad \rho_{1,2} = \frac{-r \pm \sqrt{r^2 - 4\lambda c^2}}{2}$$

But $\displaystyle r^2 - 4\lambda c^2 = r^2 - 4\left(\frac{n\pi}{\ell}\right)^2 c^2 = r^2 - \left(\frac{2n\pi c}{\ell}\right)^2$

Since $\displaystyle r < \frac{2\pi c}{\ell} \Rightarrow r < \frac{2\pi c n}{\ell}$, $n = 1, 2, \dots$. Thus $r^2 - 4\lambda c^2 < 0$

and $\displaystyle \rho_{1,2} = \frac{-r \pm i\sqrt{\left(\frac{2n\pi c}{\ell}\right)^2 - r^2}}{2}$ and

$$T_n(t) = e^{-\frac{rt}{2}}\left(A_n \cos\left(\frac{\sqrt{\left(\frac{2n\pi c}{\ell}\right)^2 - r^2}}{2}\,t\right) + B_n \sin\left(\frac{\sqrt{\left(\frac{2n\pi c}{\ell}\right)^2 - r^2}}{2}\,t\right)\right)$$

Thus $u(x,t) = e^{-rt/2} \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{\sqrt{\left(\frac{2n\pi c}{\ell}\right)^2 - r^2}}{2} t \right) + B_n \sin\left(\frac{\sqrt{\left(\frac{2n\pi c}{\ell}\right)^2 - r^2}}{2} t \right) \right] \sin \frac{n\pi x}{\ell}$

4.2 #1 Solve $u_t = k u_{xx} \quad 0 < x < \ell$

$u(0,t) = u_x(\ell, t) = 0$

$u(x,t) = X(x)\, T(t)$ or $-\frac{X''}{X} = -\frac{T'}{kT} = \lambda$ with $\lambda \geq 0$.

Thus $X'' + \lambda X = 0$

$X(0) = 0, \; X'(\ell) = 0.$

Let $\lambda = \beta^2$ then

$X(x) = A \cos \beta x + B \sin \beta x \quad , \quad 0 = X(0) = A.$

Thus $X(x) = B \sin \beta x, \quad X'(x) = B\beta \cos \beta x$ and

$0 = X'(\ell) = B\beta \cos \beta \ell.$ Thus $\cos(\beta \ell) = 0 \quad \beta \ell = n\pi + \frac{\pi}{2}, \; n = 0, 1, 2$

$\Rightarrow \beta = \left(n + \frac{1}{2}\right) \frac{\pi}{\ell}$

If $\lambda = 0$, $X''(x) = 0 \Rightarrow X(x) = Ax + B$, $X(0) = B = 0$

Thus $X(x) = Ax$ and $X'(x) = A$. $X'(\ell) = A = 0$. Thus $\lambda \neq 0$.

Then eigenvalues are $\lambda = \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{\ell^2}$ and $X_n(x) = \sin\left(\left(n + \frac{1}{2}\right) \frac{\pi x}{\ell} \right)$ for $n = 0, 1, 2$.

In addition $T_n(t) = e^{-\lambda k t} = e^{-\left(n + \frac{1}{2}\right)^2 \frac{\pi^2 k}{\ell^2} t}$

and
$$u(x,t) = \sum_{n=0}^{\infty} e^{-\left(n+\frac{1}{2}\right)^2 \frac{\pi^2 k t}{\ell^2}} A_n \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{\ell}\right) \quad , 0 < x < \ell$$

#2 $\quad u_{tt} - c^2 u_{xx} = 0 \quad , 0 < x < \ell$

$u_x(0,t) = 0, \; u(\ell, 0) = 0$

$$-\frac{X''}{X} = -\frac{T''}{c^2 T} = \lambda \geq 0$$

$X'' + \lambda X = 0$ $\qquad$ If $\lambda = 0$, $X(x) = Ax + B$

$X'(0) = 0, \; X(\ell) = 0$ $\qquad$ $X'(x) = A = 0$.

Thus $X(x) = B$. Since $X(\ell) = 0$, $B = 0$ and $\lambda \neq 0$.

Then $X(x) = A\cos\beta x + B\sin\beta x$ for $\lambda = \beta^2 > 0$

$X'(x) = -A\beta\sin\beta x + B\beta\cos\beta x$

$X'(0) = B\beta = 0 \Rightarrow B = 0$.

Thus $X(x) = A\cos\beta x$, $X(\ell) = 0 \Rightarrow \cos\beta\ell = 0$

or $\beta = \left(n+\frac{1}{2}\right)\frac{\pi}{\ell}$, $n = 0, 1, 2$ and $X_n(x) = \cos\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{\ell}\right)$

In addition $T'' + \beta^2 c^2 T = 0$ or

$$T(t) = A\cos(\beta c t) + B\sin\beta c t = A\cos\left(\frac{\left(n+\frac{1}{2}\right)\pi c t}{\ell}\right) + B\sin\left(\frac{\left(n+\frac{1}{2}\right)\pi c t}{\ell}\right)$$

and

$$u(x,t) = \sum_{n=0}^{\infty} \left[A_n \cos\left(\frac{(n+\frac{1}{2})\pi c t}{\ell}\right) + B_n \sin\left(\frac{(n+\frac{1}{2})\pi c t}{\ell}\right) \right] \cos\left(\frac{(n+\frac{1}{2})\pi x}{\ell}\right)$$