HW5 Solutions Chapter 3 # 78, 81, 83, 84

78 a) Let A the event that A wins and B the event that B wins.

Then P(exactly 4 games are played) = P(ABAA) + P(ABBB) + P(BAAA) + P(BABB)

Note that if A wins the first game then B has to win the second for the game to continue up to the fourth game. Thus

$P(ABAA) = P^3(A)P(B) = p^3(1-p)$ $\quad P(ABBB) = p(1-p)^3$

$P(BAAA) = p^3(1-p)$, $P(BABB) = p(1-p)^3$ and thus

$$P = 2p^3(1-p) + 2p(1-p)^3 = 2p(1-p)\left[p^2 + (1-p)^2\right] = 2p(1-p)(1-2p+2p^2)$$

b) We condition on the first two games. Let E the event that A wins the series.

Then $P(E) = P(E|AA)P(AA) + P(E|AB)P(AB) + P(E|BA)P(BA) + P(E|BB)P(BB)$

$= P(E|AA)p^2 + P(E|AB)p(1-p) + P(E|BA)p(1-p) + P(E|BB)(1-p)^2$

But $P(E|AA) = 1$ since if A wins the first 2, the winner of the series is A. It is now clear that $P(E|BB) = 0$.

Now assume that the first game is won by A and the second by B. The probability that A will win is again P(E) since the probability is independent on the event AB.

Thus $P(E|AB) = P(E|BA) = P(E)$ and

$$P(E) = p^2 + 2p(1-p)P(E) \implies P(E) = \frac{p^2}{1-2p(1-p)}$$

#81 The investor starts with \$25. If the stock goes down to \$10 she sells and looses \$15. If the stock goes up to \$40 she sells and wins \$15.

$p = 0.55$, $q = 0.45$.

This is the same as the gamble's ruin problem where she starts with 15 units and the probability that she wins the other 15 is

$$P = \frac{1-(q/p)^{15}}{1-(q/p)^{30}} = \frac{1-\left(\frac{9}{11}\right)^{15}}{1-\left(\frac{9}{11}\right)^{30}}$$

#83 a) Conditioning on the coin flip (Heads or Tails)

$$P(A) = P(A|H)P(H) + P(A|T)P(T) = \frac{4}{6}\frac{1}{2} + \frac{2}{6}\frac{1}{2} = \frac{1}{2}$$

b) $$P(R_3|R_1R_2) = \frac{P(R_1R_2R_3)}{P(R_1R_2)}$$

Conditioning on the coin

$$P(R_1R_2) = P(R_1R_2|H)P(H) + P(R_1R_2|T)P(T) = \left(\frac{2}{3}\right)^2\frac{1}{2} + \left(\frac{1}{3}\right)^2\frac{1}{2}$$

$$P(R_1R_2R_3) = P(R_1R_2R_3|H)P(H) + P(R_1R_2R_3|T)P(H) = \left(\frac{2}{3}\right)^3\frac{1}{2} + \left(\frac{1}{3}\right)^3\frac{1}{2}$$

Thus $$P(R_3|R_1R_2) = \frac{3}{5}$$

Let A the event that die A was used

c) $P(A|R_1R_2) = \dfrac{P(AR_1R_2)}{P(R_1R_2)} = \dfrac{P(R_1R_2|A)\,P(A)}{P(R_1R_2)}$

But $P(R_1R_2|A) = \left(\frac{2}{3}\right)^2$ $\quad P(A) = \frac{1}{2}$ and $P(R_1R_2) = \frac{3}{5}$

Thus $P(A|R_1R_2) = \dfrac{\frac{1}{2}\left(\frac{2}{3}\right)^2}{5/18} = \dfrac{4}{5}$.

#84 a) $P(A \text{ win}) = \sum_{i=0}^{\infty} P(\text{the first white appears on draw number } 3i+1)$

We replace the ball after each draw.

But $P(\text{First white ball appears at } 3i+1) = P(A_1 A_2 \ldots A_{3i} B)$ where A_1 the event that a non-white ball is drawn at the first draw, ..., and B the event that a white ball is drawn.

Since $P(A_i) = \frac{8}{12} = 2/3$ and $P(B) = \frac{4}{12} = \frac{1}{3}$ we have

$$P(A|\text{win}) = \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{3i} \frac{1}{3} = \frac{1}{3} \sum_{i=0}^{\infty} \left(\frac{8}{27}\right)^i = \frac{1}{3} \frac{1}{1-\frac{8}{27}}$$

$$P(B|\text{win}) = \sum_{i=0}^{\infty} P(\text{first white appears on draw number } 3i+2)$$

$$= \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{3i+1} \frac{1}{3} = \frac{2}{9} \sum_{i=0}^{\infty} \left(\frac{8}{27}\right)^i = \frac{2}{9} \frac{1}{1-\frac{8}{27}}$$

$$P(C|\text{win}) = \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{3i+2} \frac{1}{3} = \frac{4}{27} \frac{1}{1-\frac{8}{27}}$$

b) The balls are not replaced. Notice that at most 9 balls will be drawn.

Thus $P(A \text{ win}) = P(A \text{ draws the white ball on the first draw}) + P(\text{on the fourth draw})$

$+ P(\text{on the seventh draw}) = \frac{4}{12} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{4}{9} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{4}{6}$

Similarly

$$P(B \text{ win}) = \frac{8}{12}\frac{4}{11} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{4}{5}$$

$$P(C \text{ win}) = \frac{8}{12}\frac{7}{11}\frac{4}{10} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{4}{7} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{1}{5}\frac{4}{4}$$

4.1

W W W W
W W W W
B B B B
O O

2 balls chosen

If X denote our winnings then X is a random variable with values $-2, -1, 0, 1, 2, 4$

$$P(X=0) = P(2 \text{ orange selected}) = \frac{\binom{2}{2}}{\binom{14}{2}} = \frac{1}{91}$$

$$P(X=2) = P(\text{black and orange}) = \frac{\binom{4}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{8}{91}$$

$$P(X=4) = P(2 \text{ black}) = \frac{\binom{4}{2}}{\binom{14}{2}} = \frac{6}{91}$$

$$P(X=-1) = P(\text{white and orange}) = \frac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{16}{91}$$

$$P(X=-2) = P(2 \text{ white}) = \frac{\binom{8}{2}}{\binom{14}{2}} = \frac{28}{91}$$

$$P(X=1) = P(\text{black and white}) = \frac{\binom{8}{1}\binom{4}{1}}{\binom{14}{2}} = \frac{32}{91}$$

4.5 The possible values are $n, n-2, n-4, \ldots, -n+4, -n+2, -n$

4.6 The possible values are $3, 1, -1, -3$

$$P(1) = P(X=1) = P(\#\text{heads} - \#\text{tails} = 1) = P(HHT) + P(THH) + P(HTH) = \frac{3}{8}$$

$$P(X=-1) = P(HTT) + P(THT) + P(TTH) = \frac{3}{8}$$

$$P(X=3) = P(HHH) = \frac{1}{8} \qquad P(X=-3) = P(TTT) = \frac{1}{8}$$

4.8 a) The maximum value to appear in the two rolls.

X can take the values $1, 2, 3, 4, 5, 6$.

$$P(X=1) = \frac{1}{6}\frac{1}{6} = \frac{1}{36} \qquad P(X=2) = P((1,2),(2,1),(2,2)) = \frac{3}{36}$$

$$P(X=3) = P((1,3),(3,1),(2,3),(3,2),(3,3)) = \frac{5}{36}$$

$$P(X=4) = \frac{7}{36}, \quad P(X=5) = \frac{9}{36}, \quad P(X=6) = \frac{11}{36}$$

b) The minimum value to appear in the two rolls

X can take the values $1, 2, 3, 4, 5, 6$

$$P(X=1) = P\big((1,2),(1,3),(1,4),(1,5),(1,6),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1)\big)$$

$$= \frac{11}{36}$$

$$P(X=2) = \frac{9}{36} \qquad P(X=3) = \frac{7}{36} \qquad P(X=4) = \frac{5}{36} \qquad P(X=5) = \frac{3}{36}$$

$$P(X=6) = P\big((6,6)\big) = \frac{1}{36}$$

c) The sum of the two rolls. Then X can take the values $2, 3, 4, \dots, 12$

For example $P(X=5) = P\big((2,3),(3,2),(1,4),(4,1)\big) = \frac{4}{36}$ and similarly for the other probabilities.

d) The value of the first roll minus the value of the second. The random variable can take the values $-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$

For example $P(X=-3) = P\big((1,4),(2,5),(3,6)\big) = \frac{3}{36}$

(4.17) $$P(X=1) = P(X \le 1) - P(X<1) = F(1) - \lim_{n\to\infty} F\left(1-\frac{1}{n}\right)$$

$$= \frac{1}{2} + \frac{1-1}{4} - \lim_{n\to\infty} \frac{1-\frac{1}{n}}{4} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$P(X=2) = P(X\le 2) - P(X<2) = F(2) - \lim_{n\to\infty} F\left(2-\frac{1}{n}\right)$$

$$= \frac{11}{12} - \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1-\frac{1}{n}}{4}\right) = \frac{11}{12} - \frac{3}{4} = \frac{1}{6}$$

$$P(X=3) = F(3) - \lim_{n\to\infty} F\left(3-\frac{1}{n}\right) = 1 - \frac{11}{12} = \frac{1}{12}$$

b) $$P\left(\frac{1}{2} < X < \frac{3}{2}\right) = \lim_{n\to\infty} F\left(\frac{3}{2}-\frac{1}{n}\right) - F\left(\frac{1}{2}\right) =$$

$$\lim_{n\to\infty}\left(\frac{1}{2} + \frac{\frac{1}{2}-\frac{1}{n}}{4}\right) - \frac{1}{8} = \frac{1}{2}$$

(4.21) a) $E(X)$ is larger than $E(Y)$ because the random selection of students favors larger busloads.

b) $$E(Y) = 40\,P(Y=40) + 33\,P(Y=33) + 25\,P(Y=25) + 50\,P(Y=50)$$

$$= \frac{40}{4} + \frac{33}{4} + \frac{25}{4} + \frac{50}{4} = \frac{148}{4} = 37$$

$$E(X) = 40\,\frac{40}{148} + 33\,\frac{33}{148} + 25\,\frac{25}{148} + 50\,\frac{50}{148} = 39.3$$

(4.30) Let X denote the player's winnings.

$$E(X) = \sum i\,P(X=i) = \sum_{j=1}^{\infty} 2^j\,P(X=2^j) = \sum_{j=1}^{\infty} 2^j \frac{1}{2^j} = \sum_{j=1}^{\infty} 1 = \infty$$

a) No

b) Yes

In the long run we expect to earn, on average, more than 1 million, per game played. When the average does go over \$1 million we can stop and we will have won money.

4.32 Let X be the number of tests needed for a group of ten people.

Then $X = 1$ or $X = 11$ with $P(X=1) = 0.9 \times 0.9 \times \ldots \times 0.9 = (0.9)^{10}$

$P(X=11) = 1 - P(X=1) = 1-(0.9)^{10}$

Thus $E(X) = 1 \cdot P(X=1) + 11\, P(X=11) = 7.51$

4.35 X takes the values -1 and 1.1

$P(X=-1)$ = Probability of drawing marbles of different colors

$P(X=1.1)$ = -11- of the same color.

But $P(X=1.1) = \dfrac{\binom{5}{2}+\binom{5}{2}}{\binom{10}{2}} = \dfrac{4}{9}$ and $P(X=-1) = \dfrac{5}{9}$

Thus $E(X) = -\dfrac{1}{15}$

Since $E(X^2) = \frac{4}{9}(1.1)^2 + \frac{5}{9}(-1)^2$ | $Var(X) = 1.0889$

$[E(X)]^2 = \dfrac{1}{225}$

(4.40) Recall that a probability function for i successes in n trials are $P(i) = \binom{n}{i} p^i (1-p)^{n-i}$

We want $P(X \geq 4) = P(X=4) + P(X=5)$ with $n=5$ and $p = \frac{1}{3}$

Thus $P(X \geq 4) = \binom{5}{4}\left(\frac{1}{3}\right)^4 \frac{2}{3} + \binom{5}{5}\left(\frac{1}{3}\right)^5 = \frac{11}{243}$

Thus the number of correct answers is a binomial random variable with parameters $(5, 1/3)$.