HW #6 Section 5.1 #2, #5
Section 5.2 #4
Section 5.3 #11, #12, #13

#2 $\varphi(x) = x^2$, $0 \le x \le 1$

a) x^2 has a Fourier sine series with coefficients

$$A_n = \frac{2}{1}\int_0^1 x^2 \sin(n\pi x)\,dx = 2\int_0^1 \frac{x^2}{n\pi}\left[-\cos(n\pi x)\right]'\,dx$$

$$= \frac{2}{n\pi}\left[-x^2\cos(n\pi x)\right]_0^1 + \frac{2}{n\pi}\int_0^1 2x\cos(n\pi x)\,dx$$

$$= \frac{2}{n\pi}\left(-\cos(n\pi)\right) + \frac{2}{(n\pi)^2}\,2\int_0^1 x\left[\sin(n\pi x)\right]'\,dx$$

$$= -\frac{2}{n\pi}(-1)^n + \frac{4}{(n\pi)^2}\left[x\,\cancel{\sin(n\pi x)}^{\,0}\right]_0^1 - \frac{4}{(n\pi)^2}\int_0^1 \sin(n\pi x)\,dx$$

$$= (-1)^{n+1}\frac{2}{n\pi} + \frac{4}{(n\pi)^3}\left[\cos(n\pi x)\right]_0^1 = \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{(n\pi)^3}\left[\cos(n\pi) - 1\right]$$

$$= \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{(n\pi)^3}\left[(-1)^n - 1\right]$$

Thus $x^2 \sim \sum_{n=1}^{\infty} A_n \sin(n\pi x)$

b) x^2 has a Fourier cosine series with coefficients

$$A_0 = \frac{2}{l}\int_0^l \varphi(x)\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}$$

and $A_n = 2\int_0^1 \varphi(x)\cos(n\pi x)\,dx = 2\int_0^1 x^2\cos(n\pi x)\,dx$ ②

$$= 2\int_0^1 x^2\left[\frac{\sin(n\pi x)}{n\pi}\right]' dx = \frac{2}{n\pi}\left[x^2 \cancel{\sin}^{0}(n\pi x)\right]_0^1 - \frac{2}{n\pi}\int_0^1 2x\sin(n\pi x)\,dx$$

$$= -\frac{4}{n\pi}\int_0^1 x\sin(n\pi x)\,dx = \frac{4}{(n\pi)^2}\int_0^1 x\left[\cos(n\pi x)\right]' dx$$

$$= \frac{4}{(n\pi)^2}\left[x\cos(n\pi x)\right]_0^1 - \frac{4}{(n\pi)^2}\int_0^1 \cancel{\cos}^{0}(n\pi x)\,dx$$

$$= \frac{4}{(n\pi)^2}\cos(n\pi) = \frac{4(-1)^n}{(n\pi)^2}$$ and

$$x^2 \sim \sum_{n=1}^{\infty} A_n\cos(n\pi x) + \frac{1}{2}A_0 = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{(n\pi)^2}\cos(n\pi x)$$

$$= \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x)$$

Notice that if $x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x)$ and put $x = 0$

we have that $0 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$

or $-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{\pi^2}{12}$ or $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ a result

we will see again in #5.

#5

a) We have calculated in class the sine series of $\varphi(x) = x$ and found

$$A_n = (-1)^{n+1} \frac{2l}{n\pi} \quad \text{and thus}$$

$$x = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{l}\right), \quad 0 < x < l.$$

Integrating the result term by term we have

$$\int x\,dx = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int \sin\left(\frac{n\pi x}{l}\right) dx \quad \Rightarrow$$

$$\frac{x^2}{2} = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(-\frac{l}{n\pi} \cos\left(\frac{n\pi x}{l}\right)\right) + C \quad \Rightarrow$$

$$\frac{x^2}{2} = -\frac{2l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos\left(\frac{n\pi x}{l}\right) + C$$

Since C is the first term of the cosine series of $\frac{x^2}{2}$ we have

$$= \frac{2}{l} \int_0^l \frac{x^2}{2}\,dx = \frac{1}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{l^2}{3} \quad \text{and} \quad C = \frac{A_0}{2} = \frac{l^2}{6}$$

Thus $$\frac{x^2}{2} = \frac{l^2}{} - \frac{2l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos\left(\frac{n\pi x}{l}\right)$$

b) For $x=0$ we have

$$0 = \frac{\ell^2}{6} - \frac{2\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

#4 a) Let $\varphi(x)$ is odd.

Then $\varphi(x) \sim \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{\ell}\right) + B_n\sin\left(\frac{n\pi x}{\ell}\right)$, $\quad -\ell < x < \ell$

with $A_n = \frac{1}{\ell}\int_{-\ell}^{\ell}\varphi(x)\cos\left(\frac{n\pi x}{\ell}\right)dx$, $\quad n = 0, 1, \ldots$

Since φ is odd and $\cos(x)$ even, their product is odd and

$\int_{-\ell}^{\ell}\text{odd} = 0$. Thus $A_n \equiv 0$ and

$\varphi(x) \sim \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{\ell}\right)$ and the Fourier series has only sine

terms.

b) Similarly if φ is even then $\sin x$ is odd and their product

odd.

Thus $B_n = \frac{1}{\ell}\int_{-\ell}^{\ell}\varphi(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{1}{\ell}\int_{-\ell}^{\ell}\text{odd}(x)\,dx = 0$

and $\varphi(x) \sim \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{\ell}\right)$ and the Fourier series

has only cosine terms

#11 a) I did this in class

#12

$$\int_a^b f''(x)\, g(x)\, dx = \int_a^b [f'(x)]'\, g(x)\, dx$$

$$= f'g\Big|_a^b - \int_a^b f'(x)\, g'(x)\, dx \quad \text{integration by parts.}$$

#13 In the previous exercise substitute $f(x) = X(x) = g(x)$.

where $X(x)$ solves $X''(x) + \lambda X(x) = 0$ and $X(x)\, X'(x)\Big|_a^b \leq 0$.

Then $\displaystyle\int_a^b X''(x)\, X(x)\, dx = X'(x)\, X(x)\Big|_a^b - \int_a^b [X'(x)]^2\, dx$

$$\overset{X''(x) = -\lambda X(x)}{=} \quad \int_a^b -\lambda [X(x)]^2\, dx = X'(x)\, X(x)\Big|_a^b - \int_a^b [X'(x)]^2\, dx$$

$$\Rightarrow \quad \lambda = \frac{\int_a^b [X'(x)]^2\, dx}{\int_a^b [X(x)]^2\, dx} - \frac{X'(x)\, X(x)\Big|_a^b}{\int_a^b [X(x)]^2\, dx} \geq 0$$

so the eigenvalues cannot be negative.