HW7 #3, 9, 10, 12 from Section 5.4

#3 Let $f_n(x) = \begin{cases} 0 & \text{for } x = \frac{1}{2} \\ \gamma_n & \text{for } \frac{1}{2} - \frac{1}{n} \leq x < \frac{1}{2} \\ -\gamma_n & \text{for } \frac{1}{2} < x \leq \frac{1}{2} + \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$ where $\lim_{n\to\infty} \gamma_n = \infty$.

a) As $n \to \infty$, $\frac{1}{2} - \frac{1}{n}$, $\frac{1}{2} + \frac{1}{n} \to \frac{1}{2}$ and thus

$|f_n(x) - 0| = |f_n(x)| \to 0$ as $n \to \infty$ since $f_n\left(\frac{1}{2}\right) = 0$.

b) The convergence is not uniform since

$\max_{x\in\mathbb{R}} |f_n(x) - 0| = \max_{x\in\mathbb{R}} |f_n(x)| = |\gamma_n|$. But $\lim_{n\to\infty} \gamma_n = \infty$

and thus $(\gamma_n \leq |\gamma_n|)$ $\lim_{n\to\infty} |\gamma_n| = \infty$.

Thus although $f_n(x) \to 0$ pointwise $f_n(x) \not\to 0$ uniformly

c) $$\|f_n(x) - 0\|_{L^2}^2 = \int_{-\infty}^{\infty} |f_n(x)|^2 dx = \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} |f_n(x)|^2 dx + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} |f_n(x)|^2 dx$$

$$= |\gamma_n|^2 \int_{\frac{1}{2}-\frac{1}{n}}^{1/2} dx + |\gamma_n|^2 \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} dx = \frac{|\gamma_n|^2}{n} + \frac{|\gamma_n|^2}{n} = \frac{2|\gamma_n|^2}{n}$$

If $\gamma_n = n^{1/3} \Rightarrow |\gamma_n|^2 = n^{2/3}$ and

(2)

$$\|f_n(x) - 0\|^2_{L^2} = \frac{2n^{2/3}}{n} = \frac{2}{n^{1/3}} \Rightarrow \|f_n(x) - 0\|_{L^2} = \frac{\sqrt{2}}{n^{1/6}} \to 0$$

as $n \to \infty$.

d)

But if $\gamma_n = n$ then $\|f_n(x) - 0\|^2_{L^2} = \frac{2n^2}{n} = 2n$

and thus $\lim_{n\to\infty} \|f_n(x)\|_{L^2} = \infty$.

#9 $f(x)$ defined on $(-\ell, \ell)$ satisfies $f(-\ell) = f(\ell)$

We know that $a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right) d\ell$

$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right) d\ell$

$$a_n' = \frac{1}{\ell}\int_{-\ell}^{\ell} f'(x)\cos\left(\frac{n\pi x}{\ell}\right) dx = \frac{1}{\ell}\left[f(x)\cos\left(\frac{n\pi x}{\ell}\right)\right]_{-\ell}^{\ell}$$

$$- \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\left[-\frac{n\pi}{\ell}\sin\left(\frac{n\pi x}{\ell}\right)\right] dx$$

$$= \frac{1}{\ell}\left[f(\ell)\cos(n\pi) - f(-\ell)\cos(n\pi)\right]^{0} + \frac{n\pi}{\ell}\,\frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right) dx$$

$$= \frac{n\pi}{\ell} b_n$$

$$b_n' = \frac{1}{\ell}\int_{-\ell}^{\ell} f'(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{1}{\ell}\left[f(x)\sin\left(\frac{n\pi x}{\ell}\right)\right]_{-\ell}^{\ell}$$

$$-\frac{1}{\ell}\int_{-\ell}^{\ell}\frac{n\pi}{\ell}f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx = -\frac{n\pi a_n}{\ell}$$

\# 10 $\quad a_n' = \frac{n\pi}{\ell}b_n \Rightarrow b_n = \frac{\ell}{n\pi}a_n'$

$$b_n' = -\frac{n\pi a_n}{\ell} \Rightarrow a_n = -\frac{\ell}{n\pi}b_n'$$

$$|a_n| + |b_n| = \frac{\ell}{\pi n}\left(|a_n'| + |b_n'|\right) = \frac{\ell}{n\pi}\left(\frac{1}{\ell}\left|\int_{-\ell}^{\ell} f'(x)\cos\left(\frac{n\pi x}{\ell}\right)dx\right|\right.$$

$$\left.+\frac{1}{\ell}\left|\int_{-\ell}^{\ell} f'(x)\sin\left(\frac{n\pi x}{\ell}\right)dx\right|\right)$$

Since f is $C^1([-\ell,\ell])$ we have that there exists M such that $|f'(x)| \le M$ since $|\cos(x)|, |\sin(x)| \le 1$ we have

$$|a_n| + |b_n| \le \frac{1}{n\pi}\left[\int_{-\ell}^{\ell}|f'(x)|\left|\cos\left(\frac{n\pi x}{\ell}\right)\right|dx + \int_{-\ell}^{\ell}|f'(x)|\left|\sin\left(\frac{n\pi x}{\ell}\right)\right|dx\right]$$

$$\le \frac{M2\ell}{n\pi} + \frac{M2\ell}{n\pi} = \frac{4\ell M}{\pi}\frac{1}{n} = \frac{K}{n} \quad \text{for} \quad K = \frac{4\ell M}{\pi}$$

#12 The sine series of $f(x) = x$ on $(0, \ell)$ is the example 3 on page 109. We have seen that $a_n = (-1)^{n+1} \dfrac{2\ell}{n\pi}$

and thus $$x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2\ell}{n\pi} \sin\left(\frac{n\pi x}{\ell}\right)$$

Parseval's identity reads $$\sum_{n=1}^{\infty} |a_n|^2 \int_0^{\ell} |X_n(x)|^2 dx = \int_0^{\ell} |f(x)|^2 dx$$

But $|a_n|^2 = \left(\dfrac{2\ell}{n\pi}\right)^2$, $\displaystyle\int_0^{\ell} \left|\sin\left(\frac{n\pi x}{\ell}\right)\right|^2 dx = \frac{\ell}{2}$

$\displaystyle\int_0^{\ell} x^2 dx = \frac{\ell^3}{3}$. Thus

$$\sum_{n=1}^{\infty} \frac{4\cancel{\ell^2}}{n^2\pi^2} \cdot \frac{\cancel{\ell}}{2} = \frac{\cancel{\ell^3}}{3} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$