MATH 461 : HW #7

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Problem 5.13 Let X = # minutes after 10:00 o'clock till the bus arrives

Then $f_X(x) = \begin{cases} 1/30 \; ; & 0 \leq x \leq 30 \\ 0 \; ; & \text{else} \end{cases}$ $\quad F_X(x) = \begin{cases} 0 \; ; & x < 0 \\ x/30 \; ; & 0 \leq x < 30 \\ 1 \; ; & x \geq 30 \end{cases}$

a.) Required prob. $= P(X > 10) = 1 - F_X(10) = 2/3$

b.) Required prob. $= P(X > 25 \mid X > 15) = \dfrac{1 - F_X(25)}{1 - F_X(15)} = \dfrac{30-25}{30-15} = 1/3$

Problem 5.16 Assumption → amount of rainfall in a year is independent of the amount of rainfall in another year

Let X = amount of rainfall in inches, in a year ; $X \sim N(\mu = 40, \sigma^2 = 16)$

Required probability $= \sum_{k=10}^{\infty} (1-p)^k \cdot p$ where $p = P(X > 50) = 1 - \Phi_Z\left(\frac{50-40}{4}\right)$

$= (1-p)^{10} = \{\Phi_Z(2.5)\}^{10} \simeq (0.9938)^{10}$ where $Z \sim N(0,1)$

Problem 5.18 Let $Var(X) = \sigma^2$ $\therefore$ $P(X > 9) = 0.2 \Rightarrow P\left(\frac{X-5}{\sigma} > \frac{9-5}{\sigma}\right) = 0.2$

i.e. $P(Z > 4/\sigma) = 0.2$ i.e. $P(Z \leq 4/\sigma) = 0.8$ where $Z \sim N(\mu_0, \sigma_0^2)$ i.e. $N(0,1)$

$\therefore 4/\sigma \simeq 0.84 \Rightarrow Var(X) \simeq (4/.84)^2 = 22.6757$

Problem 5.19 $P(X > c) = 0.1 \Rightarrow P\left(\frac{X-12}{2} \leq \frac{c-12}{2}\right) = 0.9$ i.e. $P\left(Z \leq \frac{c-12}{2}\right) = 0.9$

where $Z \sim N(0,1)$

$\therefore \frac{c-12}{2} \simeq 1.28 \Rightarrow c \simeq 14.56$

Problem 5.22 $W \sim N(0.9, 0.003^2)$

a.) $P(.9 - 0.005 \leq W \leq .9 + 0.005) = P(-5/3 \leq Z \leq 5/3) = 2 \cdot \Phi(5/3) - 1$

$\therefore P(|W| > 0.9 + 0.005) = 2(1 - \Phi(5/3)) = 0.095$

$\Rightarrow$ 9.5% of forgings will be defective

b.) We want $P(.9 - 0.005 \leq W \leq .9 + 0.005) = 1 - 0.01$ i.e. $P\left(\frac{-0.005}{\sigma} \leq Z \leq \frac{0.005}{\sigma}\right) = .99$

i.e. $2 \cdot \Phi(0.005/\sigma) - 1 = .99$ or $\Phi(0.005/\sigma) = 0.995$

$\Rightarrow \frac{0.005}{\sigma} \simeq 2.575 \quad \therefore \sigma \simeq 0.001942$

Problem 5.37 $f_X(x) = \begin{cases} \frac{1}{2} ; & -1 \leq x \leq 1 \\ 0 ; & \text{else} \end{cases}$ $\quad F_X(x) = \begin{cases} 0 ; & x < -1 \\ \frac{x+1}{2} ; & -1 \leq x < 1 \\ 1 ; & x \geq 1 \end{cases}$

a.) $P(|X| > 0.5) = 1 - P(|X| \leq 0.5) = 1 - P(-0.5 \leq X \leq 0.5)$
$= 1 - \{F_X(.5) - F_X(-.5)\} = 1/2$

b.) $X \in (-1, 1) \Rightarrow |X| \in [0, 1)$

For $a \in [0, 1)$ $\quad P(|X| \leq a) = P(-a \leq X \leq a) = F_X(a) - F_X(-a) = a$

$\therefore$ $F_{|X|}(x) = \begin{cases} 0 ; & x < 0 \\ x ; & 0 \leq x < 1 \\ 1 ; & x \geq 1 \end{cases} \Rightarrow f_{|X|}(x) = \begin{cases} 1 ; & 0 \leq x < 1 \\ 0 ; & \text{else} \end{cases}$

$\therefore$ $|X| \sim \text{Uniform}(0,1)$

Problem 5.39 $f_X(x) = \begin{cases} e^{-x} ; & x \geq 0 \\ 0 ; & \text{else} \end{cases}$ $\quad Y = \log X \quad ; \text{for } X \geq 0, \; -\infty < Y < \infty$

$X = e^Y \quad ; \quad dX/dY = e^Y$

$\therefore f_Y(y) = e^{-e^y} \cdot e^y \quad ; y \in \mathbb{R}$

Problem 5.40 $f_X(x) = \begin{cases} 1 ; & 0 < x < 1 \\ 0 ; & \text{else} \end{cases}$ $\quad ; \quad Y = e^X \Rightarrow$ for $X \in (0,1)$
$Y \in (1, e)$

$X = \log Y \quad \therefore dX/dY = 1/Y \quad \Rightarrow f_Y(y) = \begin{cases} 1/y ; & 1 < y < e \\ 0 ; & \text{else} \end{cases}$

Problem 5.41 $R = A \cdot \sin\Theta \quad ; \quad f_\Theta(\alpha) = \begin{cases} 1/\pi ; & -\pi/2 < \alpha < \pi/2 \\ 0 ; & \text{else} \end{cases}$

$\therefore R \in (-A, A)$

$\Theta = \sin^{-1}(R/A) \Rightarrow |d\Theta/dR| = 1/\sqrt{A^2 - R^2} \Rightarrow f_R(r) = \begin{cases} 1/\pi\sqrt{A^2 - r^2} ; & -A < r < A \\ 0 ; & \text{else} \end{cases}$

Problem 6.2 a.) $p_{X_1,X_2}(0,0) = \frac{8}{13} \cdot \frac{7}{12} = 14/39 \quad ; \quad p_{X_1,X_2}(0,1) = p_{X_1,X_2}(1,0) = \frac{8 \times 5}{13 \times 12} = 10/39$

$p_{X_1,X_2}(1,1) = \frac{5}{13} \cdot \frac{4}{12} = 5/39$

b.) Let $p(a,b,c) = p_{X_1,X_2,X_3}(X_1 = a, X_2 = b, X_3 = c) \quad ; \quad a,b,c \in \{0,1\}$

Then $p(0,0,0) = \frac{8 \times 7 \times 6}{13 \times 12 \times 11} = 28/143 \quad ; \quad p(0,0,1) = p(0,1,0) = p(1,0,0) = \frac{8 \times 7 \times 5}{13 \times 12 \times 11} = 70/429$

$p(0,1,1) = p(1,0,1) = p(1,1,0) = \frac{8 \times 5 \times 4}{13 \times 12 \times 11} = 40/429$

$p(1,1,1) = \frac{5 \times 4 \times 3}{13 \times 12 \times 11} = 5/143$

$\boxed{\text{Problem 6.7}}$ $p_{X_1,X_2}(X_1=i, X_2=j) = (1-p)^i \cdot p \cdot (1-p)^j \cdot p$ ∵ trials are independent

$= p^2(1-p)^{i+j}$; $i, j = 0, 1, 2, \dots$

$\boxed{\text{Problem 6.8}}$ a.) $\int_0^\infty \int_{-y}^{y} c(y^2-x^2)e^{-y}\,dx\,dy = 1$ ∴ $c^{-1} = \int_0^\infty y^2 \cdot e^{-y} \cdot 2y - \frac{2}{3}y^3 e^{-y} \cdot dy$

$= \int_0^\infty \frac{4}{3}y^3 \cdot e^{-y}\,dy = \frac{4}{3}\int_0^{-\infty} t^3 e^t\,dt = \frac{4}{3}\left[(t^3-3t^2+6t-6)e^t\right]_0^{-\infty} = \frac{4}{3}\times 6 = 8$ ∴ $\boxed{c = 1/8}$

b.) $f_{X,Y}(x,y) = \frac{1}{8}(y^2-x^2)e^{-y}$; $-y \le x \le y$, $0 \le y < \infty$

$$f_Y(y) = \begin{cases} \int_{-y}^{y} \frac{e^{-y}}{8}(y^2-x^2)\,dx = \frac{e^{-y}}{8}\left[2y^3 - \frac{2}{3}y^3\right] = \frac{1}{6}y^3 e^{-y} & ;\ 0 < y < \infty \\ 0 & ;\ \text{else} \end{cases}$$

$$f_X(x) = \begin{cases} \int_{-x}^{\infty} \frac{e^{-y}}{8}(y^2-x^2)\,dy & ;\ x < 0 \\ \int_{x}^{\infty} \frac{e^{-y}}{8}(y^2-x^2)\,dy & ;\ x > 0 \end{cases} \equiv \begin{cases} \frac{(1-x)}{4}e^{x} & ;\ x < 0 \\ \frac{(1+x)}{4}e^{-x} & ;\ x > 0 \end{cases}$$

∴ $f_X(x) = \frac{1+|x|}{4} \cdot e^{-|x|}$; $x \in \mathbb{R}$

c.) $E\{X\} = \int_{-\infty}^{0} \frac{x(1-x)}{4}e^{x}\,dx + \int_0^\infty \frac{x(1+x)}{4}e^{-x}\,dx = \int_\infty^0 \frac{t(1+t)e^{-t}}{4}\,dt + \int_0^\infty \frac{x(1+x)e^{-x}}{4}\,dx = 0$

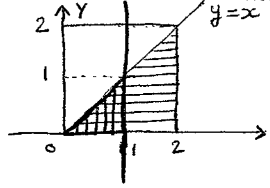
∴ $E\{X\} = 0$

$\boxed{\text{Problem 6.9}}$ a.) $\int_0^1 \int_0^2 \frac{6}{7}\left(x^2+\frac{xy}{2}\right)dy\,dx = \int_0^1 \frac{6}{7}\left[2x^2+x\right]dx = \frac{12}{7}\times\frac{1}{3} + \frac{6}{7}\times\frac{1}{2} = \frac{4}{7}+\frac{3}{7} = 1$

∴ $f(x,y)$ is indeed a joint density function.

b.) $$f_X(x) = \begin{cases} \int_0^2 \frac{6}{7}(x^2 + xy/2)\,dy = \frac{12x^2+6x}{7} & ;\ 0 < x < 1 \\ 0 & ;\ \text{else} \end{cases}$$

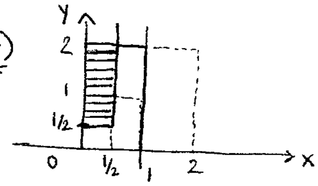
c.)



$$P(X>Y) = P(\text{shaded region})$$

$$= \int_0^1\int_0^x \frac{6}{7}\left(x^2+\frac{xy}{2}\right)dy\,dx = \int_0^1 \frac{6}{7}\left[x^3 + x^3/4\right]dx = \frac{5}{4}\times\frac{6}{7}\int_0^1 x^3\,dx = \frac{30}{28\times 4} = \frac{15}{56}$$

d.)



$$P(Y>1/2 \mid X<1/2) = \frac{P(Y>1/2,\ X<1/2)}{P(X<1/2)}$$

$$= \int_{1/2}^{2}\int_0^{1/2} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)dx\,dy \Big/ \int_0^2\int_0^{1/2}\frac{6}{7}\left(x^2+\frac{xy}{2}\right)dx\,dy$$

$$= \int_{1/2}^{2}\frac{6}{7}\left[\frac{x^3}{3}+\frac{y}{4}x^2\right]_0^{1/2}dy \Big/ \int_0^2 \frac{6}{7}\left[\frac{x^3}{3}+\frac{y}{4}x^2\right]_0^{1/2}dy = \int_{1/2}^{2}\left(\frac{1}{24}+\frac{y}{16}\right)dy \Big/ \int_0^2\left(\frac{1}{24}+y/16\right)dy$$

$$= \frac{1}{16}+\frac{1}{32}\left(4-1/4\right) \Big/ \frac{1}{12}+\frac{1}{32}(4-0) = 2+\frac{15}{4} \Big/ \frac{8}{3}+4 = 0.8625$$

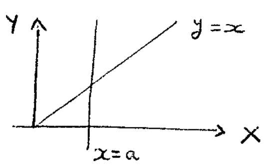
e.) $f_X(x) = \int_0^2 \frac{6}{7}\left(x^2+\frac{xy}{2}\right)dy = \frac{12x^2+6x}{7}$; $0<x<1$

$$\therefore E\{X\} = \frac{1}{7}\int_0^1 (12x^3+6x^2)\,dx = 5/7$$

f.) $f_Y(y) = \int_0^1 \frac{6}{7}\left(x^2+\frac{xy}{2}\right)dx = \dfrac{2+\frac{3y}{2}}{7}$; $0<y<2$

$$\Rightarrow E\{Y\} = \int_0^2 \left(\frac{4+3y}{14}\right)\cdot y\cdot dy = \frac{8}{14}+\frac{8}{14} = 8/7$$

Problem 6.10



a.) $P(X<Y) = \int_0^\infty\int_x^\infty e^{-y-x}\,dy\,dx = \int_0^\infty e^{-x}\cdot e^{-x}\,dx = 1/2$

b.) $P(X<a) = \int_0^a\int_0^\infty e^{-y-x}\,dy\,dx = \int_0^a e^{-x}\,dx = 1-e^{-a}$