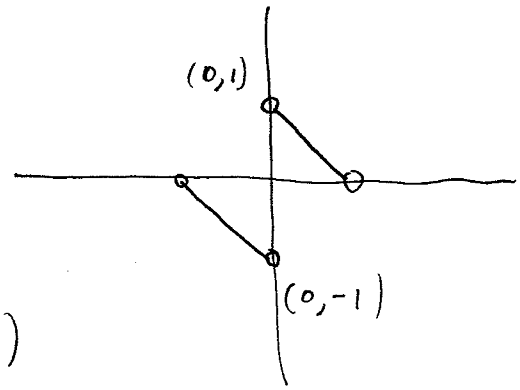
HW #8 Section 5.4 #7, Section 5.5 #2, 3, 5, 7

#7 $\varphi(x) = \begin{cases} -1-x, & -1<x<0 \\ 1-x & 0<x<1 \end{cases}$



a) $\varphi(x) \sim \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) + B_n \sin(n\pi x)$

where $A_0 = \int_{-1}^{1} \varphi(x)\,dx$, $A_n = \int_{-1}^{1} \varphi(x)\cos(n\pi x)\,dx$

$B_n = \int_{-1}^{1} \varphi(x)\sin(n\pi x)\,dx$

$\varphi(x)$ is odd and thus $\varphi(x)\cos(n\pi x)$ is odd and

$A_n = \int_{-1}^{1} \text{odd}(x)\,dx = 0$. Thus $\underline{A_n = 0}$, $n = 0, 1, 2$

$$B_n = \int_{-1}^{1} \varphi(x)\sin(n\pi x)\,dx = \int_{-1}^{0} -(1+x)\sin(n\pi x)\,dx + \int_{0}^{1} (1-x)\sin(n\pi x)\,dx$$

or

$$= 2\int_0^1 \varphi(x)\sin(n\pi x)\,dx = 2\int_0^1 (1-x)\sin(n\pi x)\,dx$$

$$= 2\int_0^1 \sin(n\pi x)\,dx - 2\int_0^1 x\sin(n\pi x)\,dx$$

$= \ldots = \frac{2}{n\pi}$. Thus $\varphi(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi x)$

or $$\varphi(x) \sim \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n}, \quad x \in (-1,1)$$

b) $$\frac{2}{\pi}\sum_{n=1}^{3} \frac{\sin(n\pi x)}{n} = \frac{2}{\pi}\sin(\pi x) + \frac{1}{\pi}\sin(2\pi x) + \frac{2}{3\pi}\sin(3\pi x)$$

Thus $B_1 = \frac{2}{\pi}$, $B_2 = \frac{1}{\pi}$, $B_3 = \frac{2}{3\pi}$

c) $$\int_{-1}^{1} \varphi^2(x)\,dx \le \int_{-1}^{1} M\,dx = 2M < \infty$$

where $M = 1$ since $|\varphi(x)| \le 1$ (look at the graph on page 1)

Thus $\varphi \in L^2$ and $$\int_{-1}^{1} \left| \frac{2}{\pi}\sum_{n=1}^{N} \frac{\sin(n\pi x)}{n} - \varphi(x) \right|^2 dx \to 0$$

as $N \to \infty$. Yes

d) $\varphi(x)$ and $\varphi'(x)$ are piecewise continuous. Thus

$$\frac{2}{\pi}\sum_{n=1}^{N} \frac{\sin(n\pi x)}{n} \xrightarrow{N\to\infty} \varphi(x) \text{ pointwise for every } x \in (-1,0)\cup(0,1)$$

At $x=0$ the partial sums are zero and $\frac{\varphi(0_-)+\varphi(0_+)}{2} = \frac{1-1}{2} = 0$

Yes we have pointwise convergence on $(-1,1)$.

e) The answer is no. Look at the Gibbs phenomenon section for a discussion of this issue.

#2 We assume real-valued functions f, g.

Let $\varphi(t) = \|f + tg\|_{L^2}^2 \geq 0$ and $\langle f, g \rangle = \int f g \, dx$

We have $\varphi(t) = \int (f+tg)^2 dx = \int f^2 dx + t^2 \int g^2 dx + 2t \int f g \, dx$

The coefficient of t^2 is positive and the only critical point is at $\varphi'(t_0) = 0 \iff 2 t_0 \int g^2 dx + 2 \int f g \, dx = 0$

$\iff t_0 = - \dfrac{\int fg}{\|g\|_{L^2}^2}$. Since $\varphi''(t_0) = 2\|g\|_{L^2}^2 > 0$

t_0 is a minimum. Thus $\varphi(t) \geq \varphi(t_0) \geq 0$

But $\varphi(t_0) \geq 0 \iff \|f\|_{L^2}^2 + \dfrac{\left(\int fg\right)^2}{\|g\|_{L^2}^2} - \dfrac{2\left(\int fg\right)^2}{\|g\|_{L^2}^2} \geq 0$

or $\left(\int fg\right)^2 \leq \|f\|_{L^2}^2 \|g\|_{L^2}^2 \iff |\langle f, g \rangle| \leq \|f\| \|g\|$.

We have assumed that $g \neq 0$. If $g = 0$ the inequality is trivial.

#3 $\left| \int_0^l f(x) g(x) dx \right| \leq \left[\int_0^l f^2(x) \, dx \right]^{1/2} \left[\int_0^l g^2(x) \, dx \right]^{1/2}$

Let $f(x) \equiv f'(x)$, $g(x) = 1$ then $\left(\int_0^l f'(x) dx \right)^2 \leq \int_0^l [f'(x)]^2 dx \int_0^l dx$

$$\left[f(\ell)-f(0)\right]^2 \le \ell \int_0^{\ell} \left[f'(x)\right]^2 dx$$

#5 First we prove $\displaystyle\sum_{n=1}^{N} a_n b_n \le \left(\sum_{n=1}^{N} a_n^2\right)^{1/2} \left(\sum_{n=1}^{N} b_n^2\right)^{1/2}$

like #2 by considering $\displaystyle\sum_{n=1}^{N} (a_n + t b_n)^2$.

Then $\displaystyle\sum_{n=1}^{N} a_n b_n \le \left(\sum_{n=1}^{N} a_n^2\right)^{1/2} \left(\sum_{n=1}^{N} b_n^2\right)^{1/2} \le \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2} \left(\sum_{n=1}^{\infty} b_n^2\right)^{1/2}$

Taking the limit on the left as $N \to \infty$ we have

$$\sum a_n b_n \le \left(\sum a_n^2\right)^{1/2} \left(\sum b_n^2\right)^{1/2}$$

#7 Let $\displaystyle c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$ and

$$d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-inx} dx$$

Since $f(x) = g(x)\left[e^{ix} - 1\right]$ we have

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-inx} \left(e^{ix} - 1\right) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-(n-1)ix} dx$$

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\, e^{-inx}\, dx = d_{n-1} - d_n$$

Thus $\sum_{-N}^{N} c_n = \sum_{n=-N}^{N} (d_{n-1} - d_n) = d_{-N-1} - \cancel{d_{-N}} + \cancel{.\,.} + \cancel{d_{N-1}} - d_N$

$= d_{-(N+1)} - d_N$.

But $g \in L^2$ and by Parseval's inequality

$\sum_{n=-\infty}^{\infty} |d_n|^2 < \infty$. Thus $|d_n|^2 \to 0$ as $n \to \pm\infty$

Thus $\lim_{n\to\infty} d_n$, $\lim_{n\to-\infty} d_n$ are zero and thus

$$\lim_{N\to\infty} \sum_{-N}^{N} c_n = \lim_{N\to\infty} \left(d_{-N-1} - d_N \right) = 0 - 0 = 0$$