HW 8 Solutions

Chapter 6 #14, 20, 21, 22, 23, 27, 28, 32

#14

X position of the ambulance from $x=0$

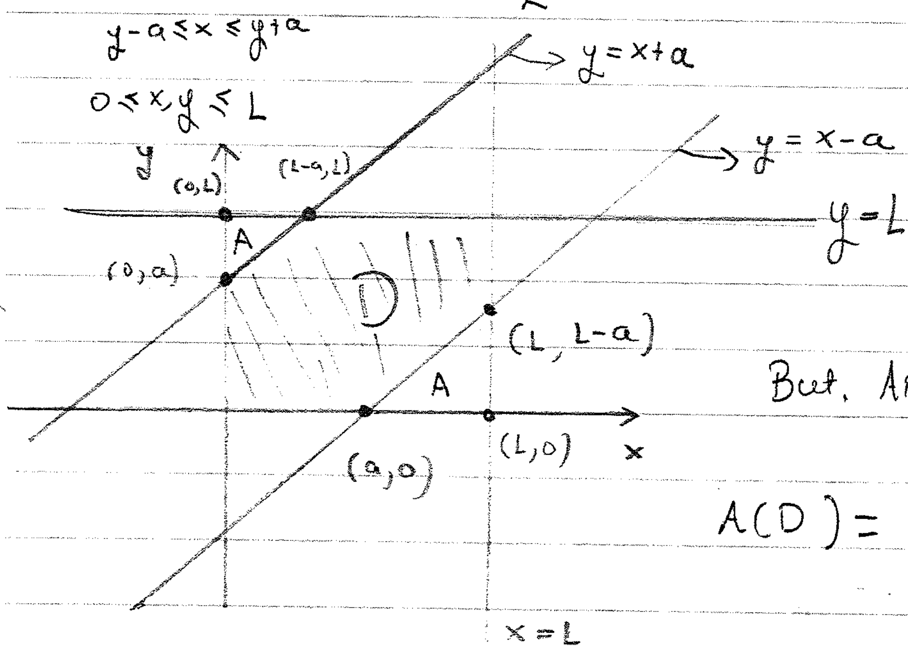
Y position of the ambulance from $x=L$.

$$f_X(x)=\begin{cases} 1/L, & 0<x<L \\ 0, & \text{otherwise}\end{cases} \qquad f_Y(y)=\begin{cases} 1/L, & 0<y<L \\ 0, & \text{otherwise}\end{cases}$$

and by independence $f(x,y)=\begin{cases} 1/L^2, & 0<x<L,\ 0<y<L \\ 0, & \text{otherwise}\end{cases}$

We want to find $F_{|X-Y|}(a)=P\{|X-Y|\leq a\}=P\{Y-a\leq X\leq Y+a\}$

$$=\iint f(x,y)\,dx\,dy=\frac{1}{L^2}\,\text{Area}(D) \quad \text{where } D=\{(x,y)\in\mathbb{R}^2:\ 0<x,y<L,\ y-a\leq x\leq y+a\}$$



$0<a\leq L$

But $A(D)=L^2-2\,\text{Area}(A)$

But $\text{Area}(A)=\frac{1}{2}(L-a)^2$ and thus

$$A(D)=2La-a^2$$

and $F_{|X-Y|}(a) = \begin{cases} \dfrac{2aL-a^2}{L^2} & , 0 \leq a \leq L \\ 0 & , \text{otherwise} \end{cases}$

and $f_{|X-Y|}(a) = \dfrac{dF}{da} = \dfrac{2}{L} - \dfrac{2a}{L^2} = \dfrac{2}{L}\left(1 - \dfrac{a}{L}\right)$

Thus $f(a) = \begin{cases} 2/L\left(1-\dfrac{a}{L}\right) & , 0 \leq a \leq L \\ 0 & , \text{otherwise} \end{cases}$

#20 a) $f(x,y) = \begin{cases} x e^{-(x+y)} & , x, y > 0 \\ 0 & , \text{otherwise} \end{cases}$

We have that $f_X(x) = \int_0^\infty f(x,y)\,dy = \int_0^\infty x e^{-x} e^{-y}\,dy = x e^{-x} \int_0^\infty e^{-y}\,dy = x e^{-x}$

$f_Y(y) = \int_0^\infty x e^{-x} e^{-y}\,dx = e^{-y} \int_0^\infty x e^{-x}\,dx$

$= e^{-y}\left(\left[-x e^{-x}\right]_0^\infty + \int_0^\infty e^{-x}\,dx \right) = e^{-y}(0+1) = e^{-y}$

Since $f_X(x) f_Y(y) = x e^{-x} e^{-y} = f(x,y)$ X, Y are independent

b) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$. But $f(x,y) = \begin{cases} 2 & \text{if } x < y < 1 \\ 0 & \text{otherwise} \end{cases}$ for each fixed x

Thus $f_X(x) = \int_x^1 2\,dy = 2(1-x)$

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^y 2\,dy = 2y$ since

$$f(x,y)=\begin{cases}2, & 0<x<y \\ 0, & \text{otherwise}\end{cases} \quad \text{for fixed } 0<y<1.$$

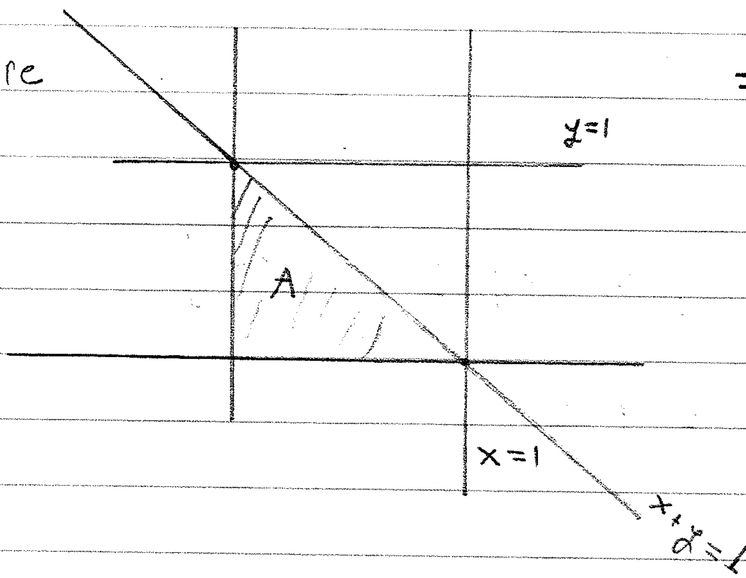
Since $f_X(x)\,f_Y(y) \neq f(x,y)$ the r.v. are not independent.

#21

$$f(x,y)=\begin{cases}24xy, & 0\le x\le 1,\ 0\le y\le 1,\ 0\le x+y\le 1 \\ 0, & \text{otherwise}\end{cases}$$

a) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \iint_A 24xy\,dx\,dy$ where A is the area in the

figure



$$= \int_0^1\int_0^{1-x} 24xy\,dy\,dx$$

$$= \int_0^1 24x\left(\int_0^{1-x} y\,dy\right)dx$$

$$= \int_0^1 12x\,\left[y^2\right]_0^{1-x}dx$$

$$= 12\int_0^1 x(1-x)^2\,dx = 12\int_0^1 (x^3+x-2x^2)\,dx = \left(\frac{x^4}{4}+\frac{x^2}{2}-\frac{2x^3}{3}\right)\Bigg|_0^1 \times 12$$

$$= \left(\frac{1}{4}+\frac{1}{2}-\frac{2}{3}\right)12 = 3+6-8 = 1$$

Thus $\iint f(x,y)\,dx\,dy = 1$ and f is a joint density

b), c) $\displaystyle f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{\{0\le y\le 1\}\cap\{0\le x+y\le 1\}} f(x,y)\,dy = \int_0^{1-x} 24xy\,dy$

$$= 12x\,y^2\Big|_0^{1-x} = 12x(1-x)^2$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{\{0\le x\le 1\}\cap\{0\le x+y\le 1\}} f(x,y)\,dx = \int_0^{1-y} 24xy\,dy = 12y(1-y)^2$$

Thus $E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_0^1 x\, 12x(1-x)^2\,dx = \frac{2}{5}$

and similarly $E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \int_0^1 12y^2(1-y)^2\,dy = \frac{2}{5}$

#22 $f(x,y) = \begin{cases} x+y, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$

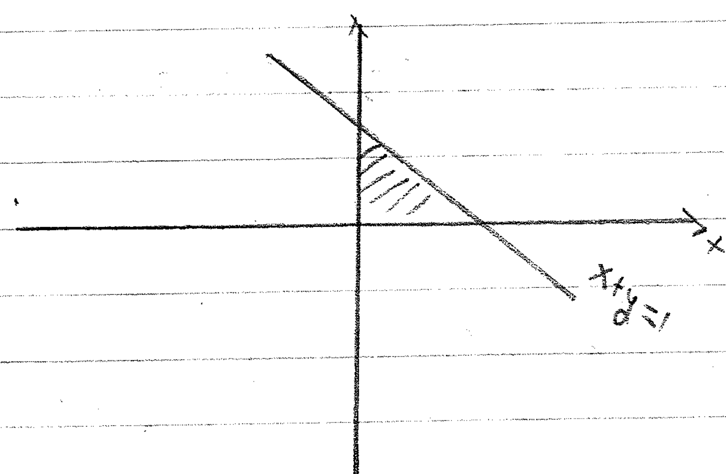
a) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (x+y)\,dy = xy + \frac{y^2}{2}\Big|_0^1 = x + \frac{1}{2}$

$f_Y(y) = \int_0^1 (x+y)\,dx = \frac{x^2}{2} + xy\Big|_0^1 = y + \frac{1}{2}$

and $f_X(x)\,f_Y(y) \ne f(x,y)$, X, Y are not independent

b) $f_X(x) = \begin{cases} x+\frac{1}{2}, & 0<x<1 \\ 0; & \text{otherwise} \end{cases}$

c) $P\{X+Y<1\} = \iint_{x+y<1} f(x,y)\,dx\,dy$



$$= \int_0^1 \int_0^{1-x} (x+y)\,dy\,dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 \left[x(1-x) + \frac{1}{2}(1-x)^2\right] dx$$

$$= \int_0^1 \left(x - x^2 + \frac{x^2}{2} + \frac{1}{2} - x\right) dx = \int_0^1 \left(\frac{1}{2} - \frac{1}{2}x^2\right) dx = \left[\frac{x}{2} - \frac{x^3}{6}\right]_0^1 = \frac{1}{3}$$

#23 $f(x,y) = \begin{cases} 12xy(1-x) & , 0<x<1, 0<y<1 \\ 0 & , \text{otherwise} \end{cases}$

a) As before $f_X(x) = 6x(1-x)$, $f_Y(y) = 2y$

Since $f_X(x) f_Y(y) = 6x(1-x)\, 2y = 12xy(1-x) = f(x,y)$ we have that X and Y are independent.

b) $E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 x\, 6x(1-x)\, dx = \frac{1}{2}$

c) $E(Y) = \int_0^1 2y^2 dy = \frac{2}{3} y^3 \Big|_0^1 = \frac{2}{3}$

d) $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^1 x^2\, 6x(1-x)\, dx = \frac{3}{10}$

and thus $Var(X) = E(X^2) - [E(X)]^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$

e) $E(Y^2) = \int_0^1 y^2\, 2y\, dy = \frac{y^4}{2}\Big|_0^1 = \frac{1}{2}$

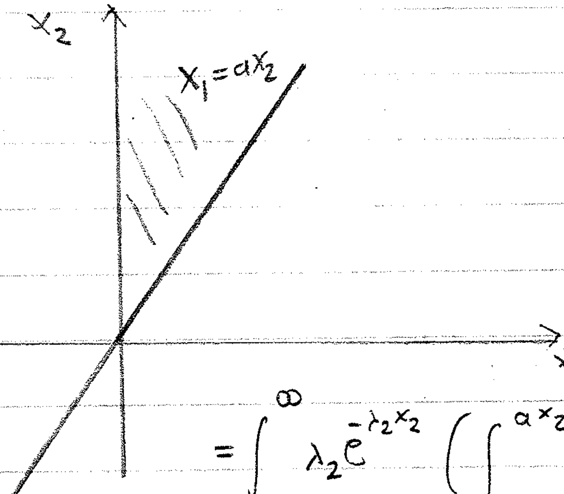
$Var(Y) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$

\# 27 Since X_1, X_2 are independent exponential random variables with λ_1 and λ_2 we have

$$f_{X_1}(x_1) = \begin{cases} \lambda_1 e^{-\lambda_1 x_1} & , x_1 \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$f_{X_2}(x_2) = \begin{cases} \lambda_2 e^{-\lambda_2 x_2} & , x_2 \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

We want $F_Z(a) = P(Z \leq a) = P\left(\frac{X_1}{X_2} \leq a\right) = \iint\limits_{x_1 \leq a x_2} f(x_1) f(x_2)\, dx_1\, dx_2$



$$\int_0^{\infty} \int_0^{a x_2} f(x_1) f(x_2)\, dx_1\, dx_2$$

$$= \int_0^{\infty} \lambda_2 e^{-\lambda_2 x_2} \left(\int_0^{a x_2} \lambda_1 e^{-\lambda_1 x_1} dx_1 \right) dx_2$$

$$= \int_0^{\infty} \lambda_2 e^{-\lambda_2 x_2} \left[-e^{-\lambda_1 x_1} \right]_0^{a x_2} dx_2 = \int_0^{\infty} \lambda_2 e^{-\lambda_2 x_2} \left(1 - e^{a \lambda_1 x_2} \right) dx_2$$

$$= \int_0^{\infty} \lambda_2 e^{-\lambda_2 x_2} dx_2 - \int_0^{\infty} \lambda_2 e^{-(\lambda_2 + a\lambda_1) x_2} dx_2 =$$

$$-e^{-\lambda_2 x_2} \Big|_0^{\infty} + \left[\frac{\lambda_2}{\lambda_2 + a\lambda_1} e^{-(\lambda_2 + a\lambda_1) x_2} \right]_0^{\infty} = 1 - \frac{\lambda_2}{\lambda_2 + a\lambda_1} = \frac{a\lambda_1}{\lambda_2 + a\lambda_1}$$

Thus $F_{\frac{X_1}{X_2}}(a) = \frac{a\lambda_1}{\lambda_2 + a\lambda_1}$ for $a > 0$

For a continuous random variable we know that $P\{X_1 = X_2\} = P\{Z = 1\} = 0$

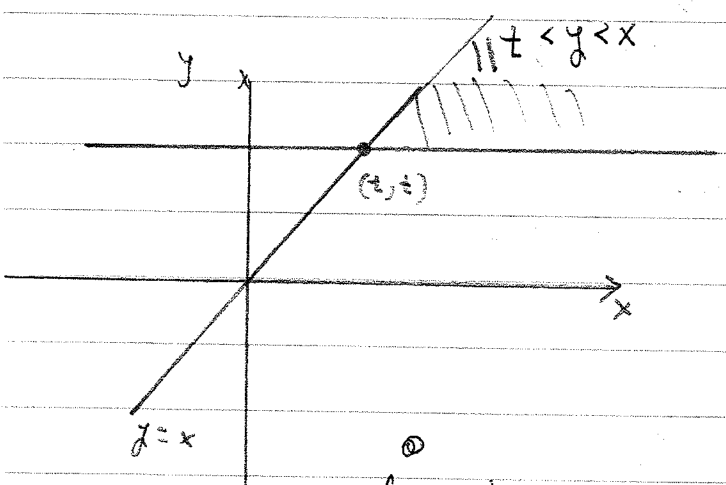
and thus $P\{X_1 < X_2\} = P\{X_1 \leq X_2\} = P\{Z \leq 1\} = F_Z(1) = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$

#28 Let X is the service time of AJ and Y is the service time of MJ.

a) We want to find $P\{Y < X \mid Y > t\}$

$$= \frac{P\{Y < X, Y > t\}}{P\{Y > t\}}$$

But $P\{Y < X, Y > t\} = \iint f(x,y)\, dx\, dy$ with $f(x,y) = f(x) f(y) = e^{-x} e^{-y}$



$$= \int_t^{\infty} \int_t^{x} e^{-x} e^{-y}\, dy\, dx$$

$$= \int_t^{\infty} e^{-x} \left(\int_t^{x} e^{-y}\, dy \right) dy = \int_t^{\infty} e^{-x} \left[-e^{-y} \right]_t^{x} dx$$

$$= \int_t^{\infty} \left(e^{-t-x} - e^{-2x} \right) dx = \left[\tfrac{1}{2} e^{-2x} - e^{-t-x} \right]_t^{\infty}$$

$$= e^{-2t} - \tfrac{1}{2} e^{-2t} = \tfrac{1}{2} e^{-2t}$$

$$P\{Y > t\} = \int_0^{\infty} \int_t^{\infty} e^{-x} e^{-y}\, dy\, dx = \int_0^{\infty} e^{-x} \left[-e^{-y} \right]_t^{\infty} dx = \int_0^{\infty} e^{-x-t}\, dX$$

$$= \left[-e^{-x-t}\right]_0^{\infty} = e^{-t}.$$

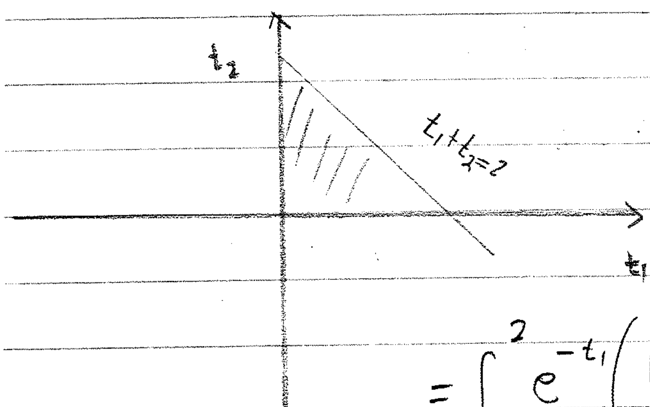
Thus $P\{Y<X \mid Y>t\} = \frac{1}{2}\frac{e^{-2t}}{e^{-t}} = \frac{1}{2}e^{-t}$.

b) Let T the total time of service of MJ's car.

Since $T = T_1+T_2$ where T_1 is the time that it takes to service AJ's car and T_2 is the time that takes to service AJ's car.

T_1 and T_2 are described by the independent exponential random variables with joint density: $f(t_1,t_2) = e^{-t_1}e^{-t_2}$ for $0<t_1, t_2<\infty$

Thus we want $P\{t_1+t_2<2\} = \iint\limits_{t_1+t_2<2} e^{-t_1-t_2}\,dt_1\,dt_2$



$$= \int_0^2\int_0^{2-t_1} e^{-t_1}e^{-t_2}\,dt_2\,dt_1$$

$$= \int_0^2 e^{-t_1}\left(\int_0^{2-t_1} e^{-t_2}\,dt_2\right)dt_1$$

$$= \int_0^2 e^{-t_1}\left[-e^{-t_2}\right]_0^{2-t_1} dt_1 = \int_0^2 e^{-t_1}\left(1-e^{t_1-2}\right)dt_1$$

$$= \int_0^2\left(e^{-t_1}-e^{-2}\right)dt_1 = \left[-e^{-t_1}-t_1e^{-2}\right]_0^2 = -e^{-2}-2e^{-2}+1 = 1-3e^{-2}$$

#32 We have seen that the number of typographical errors on a page follows a Poisson distribution where $p(i) = e^{-\lambda} \frac{\lambda^i}{i!}$

Let X_i the number of errors on page i.

Then $X = \sum_{i=1}^{10} X_i$ is the number of errors in the 10 pages. Since each X_i is Poisson with parameter $\lambda = 0.2$ we have seen that the sum of the independent Poisson r.v. X_i is also Poisson with parameter $\lambda_1 + \dots + \lambda_{10} = 10\lambda = 10 \cdot 0{,}2 = 2$.

Thus $p_X(i) = e^{-2} \frac{2^i}{i!}$

a) $P(X=0) = p_x(0) = e^{-2}$

b) $P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1 - e^{-2} - 2e^{-2} = 1 - 3e^{-2}$