# HW #9

Section 6.1 #2, 4, 9

Section 6.3 #1

(#2) Solve $\Delta u = k^2 u$ in 3d, when the solution depends only on $r$.

$\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$ and we want to solve

$$u_{rr} + \frac{2}{r}u_r = k^2 u \ (1). \quad \text{Set } u = \frac{v}{r} \Rightarrow v = ru$$

$$v_r = ru_r + u, \quad v_{rr} = ru_{rr} + 2u_r$$

$$(1) \times r: \quad \underbrace{ru_{rr} + 2u_r}_{v_{rr}} = k^2 \underbrace{ru}_{v} \Leftrightarrow v_{rr} = k^2 v$$

and $v(r) = Ae^{kr} + Be^{-kr} \Rightarrow u = \frac{1}{r}\left(Ae^{kr} + Be^{-kr}\right)$

(#4) Solve $\Delta_3 u = 0$, $0 < a < r < b$, $\underset{r=a}{u=A}$ $\underset{r=b}{u=B}$

$$u_{rr} + \frac{2}{r}u_r = 0 \Leftrightarrow r^2 u_{rr} + 2ru_r = 0 \quad \text{or} \quad \frac{\partial}{\partial r}\left(r^2 u_r\right) = 0$$

$$r^2 u_r = C_1 \Rightarrow u_r = \frac{C_1}{r^2} \Rightarrow u = -\frac{C_1}{r} + C_2$$

$$A = -\frac{C_1}{a} + C_2$$

$$B = -\frac{C_1}{b} + C_2$$

$$u(r) = \frac{(A-B)ab}{(b-a)r} + \frac{Aa - Bb}{a-b}$$

(#9) a) By #4 we know $u = -\frac{c_1}{r} + c_2$

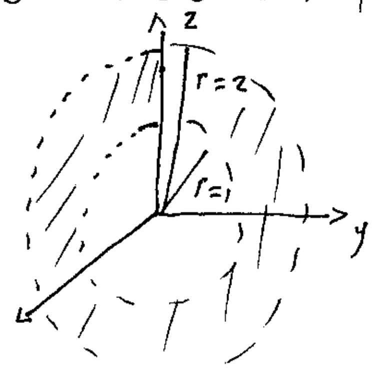
We have $\frac{\partial u}{\partial r} = \frac{c_1}{r^2}$ and $\left.\frac{\partial u}{\partial r}\right|_{r=2} = -\gamma \iff \frac{c_1}{4} = -\gamma$

$\left. u \right|_{r=1} = 100 \iff c_2 - c_1 = 100$

Thus $u(r) = \frac{4\gamma}{r} - 4\gamma + 100 \Rightarrow u(r) = 4\gamma\left(\frac{1}{r} - 1\right) + 100$

$\gamma > 0$

b) We know that the maximum and minimum value will be taken at the boundary, thus for $r=1$ and $r=2$.




For $r=1$ $u = 100$ maximum

For $r=2$ $u = 100 - 2\gamma$ minimum

c) For $u(2) = 20 \iff 4\gamma\left(\frac{1}{2} - 1\right) + 100 = 20$

$\Rightarrow \gamma = 40.$

#1 By Poisson's formula $u(x) = \frac{4 - |\vec{x}|^2}{4\pi} \int_{|\vec{x}'|=2} \frac{u(\vec{x}')}{|\vec{x} - \vec{x}'|^2} ds'$

a) By the maximum principle we know that the maximum it is taken on $\partial D$. Thus it is enough to find the maximum of

$f(\theta) = 1 + 3\sin 2\theta$ , $0 \leq \theta \leq 2\pi$

$f'(\theta) = 6\cos 2\theta = 0 \Rightarrow 2\theta = k\pi + \frac{\pi}{2}$ $\nearrow 2\theta = \frac{\pi}{2}$ $\searrow 2\theta = \frac{3\pi}{2}$

$f''(\theta) = -12\sin 2\theta$ , $f''\left(\frac{\pi}{4}\right) = -12 < 0$

$f''\left(\frac{3\pi}{4}\right) = 12 > 0$

Thus maximum at $\theta = \pi/4$ and $f\left(\frac{\pi}{4}\right) = 1 + 3\sin\left(\frac{\pi}{2}\right) = \underline{4}$

b) By the mean value property the value of $u$ at zero is equal to the average value of $u$ on the circumference of the circle $|x| = 2$.

Thus $u(0) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} (1 + 3\sin 2\theta)\,d\theta = 1.$