1 Chapter 6

41. (a) We first need to compute density functions for X and Y.

$$f_Y(y) = \int_0^\infty xe^{-x(y+1)}dx = \left[\frac{xe^{-x(y+1)}}{-(y+1)} + \frac{1}{y+1}\int e^{-x(y+1)}dx\right]_0^\infty = 0-\left(-0+\frac{1}{-(y+1)^2}\right) = \frac{1}{(y+1)^2}$$

$$f_X(x) = \int_0^\infty xe^{-x(y+1)}dy = \left[-e^{-x(y+1)}\right]_0^\infty = e^{-x}$$

Now we can compute the conditional densities.

$$f_{X|Y}(x|y) = \frac{xe^{-x(y+1)}}{\frac{1}{(y+1)^2}} = x(y+1)^2e^{-x(y+1)}$$

$$f_{Y|X}(y|x) = \frac{xe^{-x(y+1)}}{e^{-x}} = xe^{-xy+x-x} = xe^{-xy}$$

(b)

$$F_Z(a) = P(Z \le a) = P(XY \le a) = P\left(Y \le \frac{a}{X}\right) = \int_0^\infty \int_0^{\frac{a}{x}} xe^{-x(y+1)}dydx =$$

$$\int_0^\infty \left[-e^{-x(y+1)}\right]_0^{\frac{a}{x}} dx = \int_0^\infty -e^{-x(a/x)+1)} + e^{-x}dx = \left[e^{-a-x} - e^{-x}\right]_0^\infty = 1-e^{-a}$$

Now we differentiate this with respect to a to get the density.

$$f_Z(a) = \frac{d}{da}\left(1-e^{-a}\right) = e^{-a}$$

42. First we find the density of X.

$$f_X(x) = \int_{-x}^{x} c(x^2-y^2)e^{-x}dy = c\int_{-x}^{x} x^2e^{-x} - y^2e^{-x}dy = c\left[x^2ye^{-x} - \frac{1}{3}y^3e^{-x}\right]_{-x}^{x} =$$

$$c\left(x^3e^{-x} - \frac{1}{3}x^3e^{-x} - \left(-x^3e^{-x} + \frac{1}{3}x^3e^{-x}\right)\right) = \frac{4}{3}cx^3e^{-x}$$

Now we can find the conditional density of Y,

$$f_{Y|X}(y|x) = \frac{c(x^2-y^2)e^{-x}}{\frac{4}{3}cx^3e^{-x}} = \frac{3(x^2-y^2)}{4x^3}$$

Now with this, we can find the conditional density.

$$F_{Y|X}(a|x) = P(Y \le a|X=x) = \int_{-x}^{a} \frac{3(x^2-y^2)}{4x^3} = \int_{-x}^{a} \frac{3x^2}{4x^3} - \frac{3y^2}{4x^3}dy = \int_{-x}^{a} \frac{3}{4x} - \frac{3y^2}{4x^3}dy =$$

$$\frac{3}{4}\left[\frac{y}{x} - \frac{y^3}{3x^3}\right]_{-x}^{a} = \frac{3}{4}\left(\frac{a}{x} - \frac{a^3}{3x^3} - \left(-1+\frac{1}{3}\right)\right) = \frac{3a}{4x} - \frac{a^3}{4x^3} + \frac{1}{2}$$

2 Chapter 7

6. The expected value of the sum of the rolls is the same as the sum of the expected value of the rolls. Let X_i indicate the value of the *ith* roll.

$$E\left[\sum_1^{10} X_i\right] = \sum_1^{10} E[X_i] = \sum_1^{10}\left(\frac{1}{6}+\frac{2}{6}+\frac{3}{6}+\frac{4}{6}+\frac{5}{6}+\frac{6}{6}\right) = \sum_1^{10}\frac{21}{6} = \frac{210}{6} = 35$$

7. (a) We want to find the expected value of the items in both A and B. Therefore,

$$E[A\cap B] = 0 + 1\frac{\binom{10}{1}\binom{9}{2}\binom{7}{2}}{\binom{10}{3}^2} + 2\frac{\binom{10}{2}\binom{8}{1}\binom{7}{1}}{\binom{10}{3}^2} + 3\frac{\binom{10}{3}}{\binom{10}{3}^2} = 0.9$$

(b) This time we want to find the expectation for the number of objects chosen by neither A or B. Therefore,

$$E[(A\cup B)^c] = 4\frac{\binom{10}{3}\binom{7}{3}}{\binom{10}{3}^2} + 5\frac{\binom{10}{3}\binom{3}{1}\binom{7}{2}}{\binom{10}{3}^2} + 6\frac{\binom{10}{3}\binom{3}{2}\binom{7}{1}}{\binom{10}{3}^2} + 7\frac{\binom{10}{3}}{\binom{10}{3}^2} = 4.9$$

(c) Now we want to find the expectation of the number of objects found by exactly one of A and B. Therefore,

$$E[(A\cap B^c)\cup(A^c\cap B)] = 0 + 2\frac{\binom{10}{3}\binom{3}{2}\binom{7}{1}}{\binom{10}{3}^2} + 4\frac{\binom{10}{3}\binom{3}{1}\binom{7}{2}}{\binom{10}{3}^2} + 6\frac{\binom{10}{3}\binom{7}{3}}{\binom{10}{3}^2} = 4.2$$

11. Let X_i denote whether on the *ith* flip a changeover occurs. This will be a bernoulli random variable. We need to compute the probability a changeover occurs. We know this will happen if our current flip is a heads and the next is a tails or if the current is tails and the next is heads. Therefore,

$$p' = P(TH) + P(HT) = (1-p)p + p(1-p) = 2p(1-p)$$

So now we know X_i is a bernoulli random variable with $p' = 2p(1-p)$. It follows that

$$E[X] = E\left[\sum_{i=1}^{n-1} X_i\right] = \sum_{i=1}^{n-1} E[X_i] = \sum_{i=1}^{n-1} 2p(1-p) = 2p(1-p)(n-1)$$

21. (a) Let X_i be an indicator variable for the number of birthdays on a day i. We need to compute the probability that 3 people have a birthday on that given day. So we want to fix 3 people's birthdays to this day and the other 97 to one of the 364 other days. Let $X_i = 1$ if the number of birthdays on day i is exactly 3 and $X_i = 0$ otherwise. Therefore,

$$P(X_i = 1) = \frac{\binom{100}{3}364^{97}}{365^{100}}$$

$$E[X] = E\left[\sum_{i=1}^{365} X_i\right] = \sum_{i=1}^{365} E[X_i], = 365\frac{\binom{100}{3}364^{97}}{365^{100}} = \frac{\binom{100}{3}364^{97}}{365^{99}} \approx 0.93$$

(b) This time, let $X_i = 1$ if at least one person has their birthday on day i and $X_i = 0$ if no people have their birthday on day i.

$$P(X_i = 1) = 1 - \frac{364^{100}}{365^{100}}$$

$$E[X] = E\left[\sum_{i=1}^{365} X_i\right] = \sum_{i=1}^{365} E[X_i] = 365\left(1 - \left(\frac{364}{365}\right)^{100}\right) = 365 - \frac{364^{100}}{365^{99}} \approx 87.57$$

30.

$$\text{Var}(X) = \sigma^2 = E[X^2] - E[X]^2 \iff E[X^2] = \sigma^2 + \mu^2$$

$$E[(X-Y)^2] = E[X^2 - 2XY + Y^2] = E[X^2] - 2E[XY] + E[Y^2] = 2(\sigma^2 + \mu^2) - 2E[X]E[Y] = 2\sigma^2$$

31. So we want to compute the variance. Therefore,

$$\text{Var}\left(\sum_1^{10} X_i\right) = \sum_1^{10} \text{Var}(X_i) = \sum_1^{10} (E[X_i^2] - (E[X_i])^2) = \sum_1^{10}\left(\frac{91}{6} - \left(\frac{21}{6}\right)^2\right) = \frac{175}{6}$$

33. (a) We know that $E[X] = 1$ and $\text{Var}(X) = 5$. Therefore,

$$E[X^2] = \text{Var}(X) + E[X]^2 = 5 + 1^2 = 6$$

$$E[(2+X)^2] = E[4 + 4X + X^2] = 4 + 4 * 1 + 6 = 14$$

(b)

$$\text{Var}(4 + 3X) = 3^2 * \text{Var}(X) = 9 * 5 = 45$$

38. We know that $\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$. Therefore, we will compute $E[X]$ and $E[Y]$ and $E[XY]$.

$$f_X(x) = \int_0^x \frac{2e^{-2x}}{x} dy = \left[y\frac{2e^{-2x}}{x}\right]_0^x = 2e^{-2x}$$

$$E[X] = 2\int_0^\infty xe^{-2x}dx = 2\left[x\frac{-1}{2}e^{-2x} - \int \frac{-1}{2}e^{-2x}dx\right]_0^\infty = 2\left[x\frac{-1}{2}e^{-2x} - \frac{1}{4}e^{-2x}\right]_0^\infty = 2\left(0 - \frac{-1}{4}\right) = \frac{1}{2}$$

$$E[Y] = \int_0^\infty \int_0^x y\frac{2e^{-2x}}{x}dydx = \int_0^\infty \left[\frac{1}{2}y^2\frac{2e^{-2x}}{x}\right]_0^x dx = \int_0^\infty xe^{-2x}dx = \frac{1}{4}$$

$$E[XY] = \int_0^\infty \int_0^x xy\frac{2e^{-2x}}{x}dydx = \int_0^\infty \left[xy^2\frac{e^{-2x}}{x}\right]_0^x dx = \int_0^\infty x^2e^{-2x}dx = \left[x^2\frac{-1}{2}e^{-2x} - \int \frac{-1}{2}2xe^{-2x}dx\right]_0^\infty =$$

$$\left[\frac{-1}{2}x^2e^{-2x} + \left(\frac{-1}{2}xe^{-2x} - \int \frac{-1}{2}e^{-2x}dx\right)\right]_0^\infty = \left[\frac{-1}{2}x^2e^{-2x} + \left(\frac{-1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right)\right]_0^\infty =$$

$$\left[\frac{-1}{2}x^2e^{-2x} - \frac{-1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^\infty = \frac{1}{4}$$

Now we can finally compute the covariance.

$$\text{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])] = E\left[\left(X - \frac{1}{2}\right)\left(Y - \frac{1}{4}\right)\right] = E\left[XY - \frac{1}{4}X - \frac{1}{2}Y + \frac{1}{8}\right] =$$

$$E[XY] - \frac{1}{4}E[X] - \frac{1}{2}E[Y] + \frac{1}{8} = E[XY] - \frac{1}{8} - \frac{1}{8} + \frac{1}{8} = E[XY] - \frac{1}{8} = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$