SEMINAR NOTES on Hrushovski's

*Stable group theory and approximate subgroups* [3]

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For motivational background we first read two other papers:

Edgar & Miller, *Borel subrings of the reals*, PAMS 131 (2003) 1121–1129
Tao, *The sum-product phenomenon in arbitrary rings*, arXiv:0806.2497v5

Next we had talks on Hrushovski's paper above. The notes below on sections 2 and 3 are a record of talks by Henson and myself, assume only rudimentary knowledge of model theory, and give detailed proofs. If possible we choose notations and formulations similar to those in Hrushovski's paper. More to follow. Needless to say, the source for all this is [3] to which we also refer for references to other original papers and motivational comments. See also Tao [4] for discussions around [3].

Let us fix some global notational conventions used throughout:

We let $m, n$ range over $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $X, Y$ be sets and $R \subseteq X \times Y$ a relation. For $a \in X, b \in Y$ we often write $R(a, b)$ (or even $aRb$) to indicate that $(a, b) \in R$. For $a \in X$ we put

$$R(a) := \{b \in Y : \; R(a, b)\}.$$

We also let $\breve{R} := \{(b, a) \in Y \times X : (a, b) \in R\}$ be the reverse of $R$, so for $b \in Y$ we have $\breve{R}(b) = \{a \in X : \; R(a, b)\}$. Unless specified otherwise, $|X|$ is the size (cardinality) of $X$.

## 1. Independence

This section corresponds to Section 2 in [3]. Its subsections are: *Many-sorted structures*; *The monster model*; *Finite satisfiability*; *A-invariant types*; *Dividing and forking*; *Stable separation*; *Stable relations*; *The A-topology*; *Keisler measures*; *Ideals.* (For later applications the subsection on Keisler measures probably needs some additional material.)

**Many-sorted structures.** Let $L$ be an $S$-*sorted* first-order language, so an $L$-structure $\mathcal{M} = (M; \cdots)$ has a family $M = (M_s)_{s \in S}$ of underlying sets rather than a single underlying set. The set $S$ of sorts is part of what determines $L$, and we define the size $|L|$ of $L$ to be the cardinal

$$|L| := \max\{\aleph_0, \; |\text{set of nonlogical symbols of } L|, \; |S|\}.$$

For each sort $s \in S$ we have variables of sort $s$ that we think of as ranging over the underlying set $M_s$, for any $L$-structure $\mathcal{M}$ as above.

Let $\mathcal{M} = (M; \cdots)$ be an $L$-structure. Given a variable $v$ of sort $s$, we put $M_v := M_s$. An $S$-sorted multivariable is a family $x = (x_i)_{i \in I}$ where each

$x_i$ is a variable of some sort $s_i \in S$, and $x_i$ and $x_j$ are different variables (possibly of the same sort) whenever $i \neq j$; for such $x$ we set

$$M_x := \prod_i M_{x_i} \qquad \text{(the } x\text{-set of } \mathcal{M}), \qquad |x| := |I| \quad \text{(size of } x),$$

and the elements $a = (a_i) \in M_x$ are also called $x$-tuples in $\mathcal{M}$ (of size $|I|$). A *parameter set* (in $\mathcal{M}$) is a family $A = (A_s)_{s \in S}$ with $A_s \subseteq M_s$ for all $s \in S$, and has size $|A| := \sum_s |A_s|$. Let $A$ be a parameter set. For a variable $v$ of sort $s \in S$ we set $A_v := A_s$ and for an $S$-sorted multivariable $x = (x_i)$ we put $A_x := \prod_i A_{x_i}$. We have the extended language $L(A)$, and for any $S$-sorted multivariable $x$ we have the notion of an $A$-definable subset of $M_x$. Instead of "$M$-definable" we also write just "definable" when the ambient structure $\mathcal{M}$ is clear from the context. For definable $X \subseteq M_x$ we let $\neg X$ be the complement of $X$ in $M_x$, we let $\mathrm{Def}(X)$ be the boolean algebra of definable subsets of $X$, and we let $\mathrm{St}(X)$ be the Stone space of this boolean algebra, so the points of $\mathrm{St}(X)$ are the ultrafilters of $\mathrm{Def}(X)$.

Suppose $A$ is a parameter set in $\mathcal{M}$. If $X \subseteq M_x$ is $A$-definable, then $\mathrm{Def}(X|A)$ is the boolean algebra of its $A$-definable subsets, with Stone space $\mathrm{St}(X|A)$. We also set $\mathrm{St}_x(A) := \mathrm{St}(M_x|A)$, in particular, $\mathrm{St}_x(M) = \mathrm{St}(M_x)$. Points of any of these Stone spaces are often referred to as *types*.

For more on this many-sorted set-up, see Section 3 in [1], with definitions of other basic notions such as partial elementary maps, automorphisms, saturation, homogeneity, ....

**The monster model.**[1] Throughout $\mathbb{U} = (U; \dots)$ is a big ambient $L$-structure. Here "big" means that $\mathbb{U}$ comes equipped with a certain cardinal $\kappa(\mathbb{U}) > |L|$ such that $\mathbb{U}$ is $\kappa(\mathbb{U})$-saturated and strongly $\kappa(\mathbb{U})$-homogeneous. Given our ambient $\mathbb{U}$, "small" means "of size $< \kappa(\mathbb{U})$". We let $x, y, z$ be small disjoint $S$-sorted multivariables and $A, B$ small parameter sets in $\mathbb{U}$. Also, $M$ denotes the family $(M_s)$ of underlying sets of a small elementary submodel of $\mathbb{U}$, which, abusing language, we also denote by $M$. Unless we specify otherwise, "definability" is with respect to $\mathbb{U}$. For $A$-definable $X \subseteq \mathbb{U}_x$ and $a \in X$,

$$\mathrm{tp}_X(a|A) := \{P \in \mathrm{Def}(X|A) : \ a \in P\} \qquad \text{(the type of } a \text{ over } A),$$

and for $X = \mathbb{U}_x$ we drop subscript $X$ in this notation or replace it $x$; note that $\mathrm{St}(X|A) = \{\mathrm{tp}_X(a|A) : a \in X\}$.

Let $X \subseteq \mathbb{U}_x$ (not necessarily definable). We set $X(M) := X \cap M_x$, and we say that $X$ is *$A$-invariant* if $\sigma(X) = X$ for all $\sigma \in \mathrm{Aut}(\mathbb{U}|A)$. The following are equivalent:

(1) $X$ is $A$-invariant;
(2) $X$ is a union of $\mathrm{Aut}(\mathbb{U}|A)$-orbits with respect to the usual action of $\mathrm{Aut}(\mathbb{U}|A)$ on $\mathbb{U}_x$;

[1] The term "monster" does not refer here to pathology, but to being big. However, finite structures are also "big" according to the definition of that term.

(3) $X = \bigcup_{p \in E} p(\mathbb{U}_x)$ for some set $E \subseteq \mathrm{St}_x(A)$.

Let $X \subseteq \mathbb{U}_x$ be definable. A point in $\mathrm{St}(X)$ is often called a *global* type and typically denoted by a bold face letter like $\mathbf{p}$. Sometimes we want to think of $\mathbf{p}$ as the set of $L(U)$-formulas $\phi(x)$ such that $\phi(X) \in \mathbf{p}$, and in this role as a set of formulas we denote $\mathbf{p}$ by $\mathbf{p}(x)$. Likewise for types in $\mathrm{St}(X|A)$ if $X$ is $A$-definable. For more on these matters, see Section 5 of [1].

**Finite satisfiability.** Let $X \subseteq \mathbb{U}_x$. For an $L(U)$-formula $\phi(x)$ we put

$$\phi(X) := \{a \in X : \models \phi(a)\}.$$

Let $\Phi = \Phi(x)$ be a set of $L(U)$-formulas $\phi(x)$. Then

$$\Phi(X) := \{a \in X : \models \phi(a) \text{ for all } \phi \in \Phi\} = \bigcap_{\phi \in \Phi} \phi(X).$$

We say that $\Phi$ is *over* $A$ if it consists of $L(A)$-formulas. We say that $\Phi$ is *finitely satisfiable* if $\Delta(\mathbb{U}_x) \neq \emptyset$ for all finite $\Delta \subseteq \Phi$; if $\Phi$ is over $A$, this is equivalent to $\Phi(\mathbb{U}_x) \neq \emptyset$. We say that $\Phi$ is *finitely satisfiable in* $A$ if $\Delta(\mathbb{U}_x) \cap A_x \neq \emptyset$ for all finite $\Delta \subseteq \Phi$. We call $\Phi$ a *partial type* if $\mathbb{U}_x \neq \emptyset$, $\phi(\mathbb{U}_x) \neq \emptyset$ for all $\phi \in \Phi$, and $\phi_1 \wedge \phi_2 \in \Phi$ whenever $\phi_1, \ \phi_2$ in $\Phi$; note that then $\Phi$ is finitely satisfiable. A set $X \subseteq \mathbb{U}_x$ is said to *meet* $A$ if $X \cap A_x \neq \emptyset$.

Here is a useful fact about finite satisfiability in a (small) model $M$:

**Lemma 1.1.** *Suppose* $\Phi(x)$ *is finitely satisfiable in* $M$*. Then* $\Phi(x)$ *extends to a global type in* $\mathrm{St}(\mathbb{U}_x)$ *that is finitely satisfiable in* $M$*.*

*Proof.* Let $\Psi(x)$ be the set of all $L(U)$-formulas $\psi(x)$ such that $M_x \subseteq \psi(\mathbb{U}_x)$. Then $\Phi(x) \cup \Psi(x)$ is clearly finitely satisfiable (in $M$), and thus extends to a global type $\mathbf{p}(x) \in \mathrm{St}(\mathbb{U}_x)$. Then $\mathbf{p}(x)$ is finitely satisfiable in $M$: if $\theta(x)$ is an $L(U)$-formula and $\theta(\mathbb{U}_x) \cap M_x = \emptyset$, then $\neg\theta(x) \in \Psi(x) \subseteq \mathbf{p}(x)$. $\square$

**Remarks on $A$-invariant types.** Let $X \subseteq \mathbb{U}_x$ be $A$-definable (and hence $A$-invariant). Let $\mathbf{p} \in \mathrm{St}(X)$ be a global type. We consider $\mathbf{p}$ both as an ultrafilter on the boolean algebra $\mathrm{Def}(X)$, and thus as a collection of subsets of $X$, and as the set $\mathbf{p}(x)$ of $L(U)$-formulas $\phi(x)$ such that $\phi(\mathbb{U}_x) \in \mathbf{p}$. We say that $\mathbf{p}$ is *$A$-invariant* if $\sigma(P) \in \mathbf{p}$ for all $\sigma \in \mathrm{Aut}(\mathbb{U}|A)$ and all sets $P \in \mathbf{p}$. If $\mathbf{p}$ is $A$-invariant, then it is $B$-invariant for all $B \supseteq A$. Note that $\mathbf{p}(x)$ is finitely satisfiable in $A$ iff every $P \in \mathbf{p}$ meets $A$.

**Lemma 1.2.** *Suppose* $\mathbf{p}$ *is finitely satisfiable in* $A$*. Then* $\mathbf{p}$ *is $A$-invariant.*

*Proof.* Let $P \in \mathbf{p}$ and $\sigma(P) \notin \mathbf{p}$, $\sigma \in \mathrm{Aut}(\mathbb{U}|A)$. Then $P \cap \sigma(\neg P) \in \mathbf{p}$, so we can take $a \in P \cap \sigma(\neg P) \cap A_x$, so $a = \sigma(a) \in \sigma(P)$, a contradiction. $\square$

In particular, if $\Phi(x)$ is finitely satisfiable in $M$ and $\phi(\mathbb{U}_x) \subseteq X$ for all $\phi \in \Phi$, then by Lemmas 1.1 and 1.2 it extends to a global type $\mathbf{p}(x) \in \mathrm{St}(X)$ that is finitely satisfiable in $M$ and thus $M$-invariant.

**Lemma 1.3.** *Let* $\mathbf{q} \in \mathrm{St}(\mathbb{U}_y)$ *be* $A$*-invariant and* $a, b \in \mathbb{U}_x$ *with* $\mathrm{tp}(a|A) = \mathrm{tp}(b|A)$*. Suppose* $c \in \mathbb{U}_y$ *realizes* $\mathbf{q}\restriction Aa$ *and* $d \in \mathbb{U}_y$ *realizes* $\mathbf{q}\restriction Ab$*. Then* $\mathrm{tp}\big((a,c)|A\big) = \mathrm{tp}\big((b,d)|A\big)$*.*

*Proof.* Let $\phi(u,x,y)$ be an $L$-formula, $e \in A_u$. Then

$$\models \phi(e,a,c) \;\Leftrightarrow\; \phi(e,a,y) \in \mathbf{q}(y) \;\Leftrightarrow\; \phi(e,b,y) \in \mathbf{q}(y) \;\Leftrightarrow \models \phi(e,b,d).$$

□

**Lemma 1.4.** *Let* $\mathbf{q} \in \mathrm{St}(\mathbb{U}_y)$ *be* $A$*-invariant, and suppose* $(a_n)$ *and* $(b_n)$ *are sequences in* $\mathbb{U}_y$ *such that for all* $n$*,*

$$a_n \models \mathbf{q}\restriction Aa_0 \dots a_{n-1}, \qquad b_n \models \mathbf{q}\restriction Ab_0 \dots b_{n-1}.$$

*Then* $(a_n) \equiv_A (b_n)$*.*

*Proof.* Note that $\mathrm{tp}(a_0|A) = \mathrm{tp}(b_0|A) = \mathbf{q}\restriction A$. Assume inductively that

$$\mathrm{tp}((a_0,\dots,a_n)|A) = \mathrm{tp}((b_0,\dots,b_n)|A).$$

Since $a_{n+1}$ realizes $\mathbf{q}\restriction Aa_0 \dots a_n$ and $b_{n+1}$ realizes $\mathbf{q}\restriction Ab_0 \dots b_n$, it follows from the previous lemma that

$$\mathrm{tp}((a_0,\dots,a_n,a_{n+1})|A) = \mathrm{tp}((b_0,\dots,b_n,b_{n+1})|A).$$

□

With $A$-invariant $\mathbf{q} \in \mathrm{St}(\mathbb{U}_y)$, an easy recursion yields a sequence $(a_n)$ such that $a_n \models \mathbf{q}\restriction Aa_0 \dots a_{n-1}$ for all $n$, and it follows easily from Lemma 1.4, that each such sequence is indiscernible over $A$. We call such a sequence $\mathbf{q}$-*indiscernible over* $A$.

**Dividing and forking.** For $n > 1$, a collection $\mathcal{C}$ of subsets of a set $P$ is said to be $n$-*disjoint* if all sets in $\mathcal{C}$ are nonempty and for all distinct $X_1,\dots,X_n \in \mathcal{C}$ we have $X_1 \cap \dots \cap X_n = \emptyset$. A definable set $X \subseteq \mathbb{U}_x$ is said to *divide* over $A$ if $X \neq \emptyset$ and for some $n > 1$ there is an infinite $n$-disjoint collection of $A$-conjugates of $X$. As often with model-theoretic notions, it is "non-dividing" rather than "dividing" that is of most interest to us.

**Lemma 1.5.** *Let* $\phi(x,y)$ *be an* $L(A)$*-formula, and suppose* $X = \phi(\mathbb{U}_x, b)$ *is nonempty,* $b \in \mathbb{U}_y$*. Then* $X$ *divides over* $A$ *iff there is an* $A$*-indiscernible sequence* $(b_i)_{i\in\mathbb{N}}$ *in* $\mathbb{U}_y$ *with* $b_0 = b$ *and* $\bigcap_i \phi(\mathbb{U}_x, b_i) = \emptyset$*.*

*Proof.* Suppose $X$ divides over $A$. Take $n > 1$ and an $n$-disjoint infinite collection $\mathcal{C}$ of $A$-conjugates of $X$. For each $Y \in \mathcal{C}$ there is an $A$-conjugate $c$ of $b$ such that $Y = \phi(\mathbb{U}_x, c)$, so we have an infinite set $C$ of $A$-conjugages of $b$ with a bijection $c \mapsto \phi(\mathbb{U}_x, c) : C \to \mathcal{C}$. By Ramsey's theorem and saturation this yields an $A$-indiscernible sequence $(c_i)_{i\in\mathbb{N}}$ of $A$-conjugates of $b$ such that for all $i_1 < \dots < i_n$ in $\mathbb{N}$ we have

$$\phi(\mathbb{U}_x, c_{i_1}) \cap \dots \cap \phi(\mathbb{U}_x, c_{i_n}) = \emptyset.$$

It remains to apply an $A$-automorphism of $\mathbb{U}$ that sends $c_0$ to $b$.

For the converse, assume that $(b_i)_{i\in\mathbb{N}}$ is an $A$-indiscernible sequence in $\mathbb{U}_y$ with $b_0 = b$ and $\bigcap_i \phi(\mathbb{U}_x, b_i) = \emptyset$. It follows easily that $\phi(\mathbb{U}_x, b_0) \neq \phi(\mathbb{U}_x, b_1)$, and thus $\phi(\mathbb{U}_x, b_i) \neq \phi(\mathbb{U}_x, b_j)$ for all $i \neq j$. Also, by saturation there are $i_1 < \cdots < i_n$ in $\mathbb{N}$ with $n > 1$ such that

$$\phi(\mathbb{U}_x, b_{i_1}) \cap \cdots \cap \phi(\mathbb{U}_x, b_{i_n}) = \emptyset.$$

Hence $\phi(\mathbb{U}_x, b_{j_1}) \cap \cdots \cap \phi(\mathbb{U}_x, b_{j_n}) = \emptyset$ for all $j_1 < \cdots < j_n$ in $\mathbb{N}$, and thus $X$ divides over $A$. □

A definable set $P \subseteq \mathbb{U}_x$ is said to *fork over* $A$ if $P \neq \emptyset$ and there are definable sets $X_1, \ldots, X_n \subseteq \mathbb{U}_x$ that divide over $A$ such that $n \geq 1$ and

$$P \subseteq X_1 \cup \cdots \cup X_n.$$

So if a definable set $P \subseteq \mathbb{U}_x$ divides over $A$, then it forks over $A$. If a definable set $P \subseteq \mathbb{U}_x$ meets $A$, then it doesn't fork over $A$.

Let $\Phi$ be a collection of nonempty definable sets in $\mathbb{U}_x$ such that $P \cap Q \in \Phi$ for all $P, Q \in \Phi$. We say that $\Phi$ *divides* over $A$ if some set $P \in \Phi$ divides over $A$, and we say that $\Phi$ *forks* over $A$ if some set $P \in \Phi$ forks over $A$. So if $\Phi$ divides over $A$, then it forks over $A$. Of course, these definitions also apply to a partial type $\Phi(x)$ by taking $\Phi := \{\phi(\mathbb{U}_x) : \ \phi(x) \in \Phi(x)\}$. If $\Phi$ doesn't divide over $A$, then it doesn't divide over any $B \supseteq A$. If all sets in $\Phi$ meet $A$, then $\Phi$ doesn't fork over $A$.

**Lemma 1.6.** *Let $\Phi$ be as above and let $D$ be a (not necessarily small) parameter set, and suppose the sets in $\Phi$ are $D$-definable and $\Phi$ does not fork over $A$. Then $\Phi$ extends to some $p \in \mathrm{St}_x(D)$ that does not fork over $A$.*

*Proof.* Let $\Psi := \{X \in \mathrm{Def}(\mathbb{U}_x|D) : \ X \text{ forks over } A\}$. Suppose $P \in \Phi$ and $X_1, \ldots, X_n \in \Psi$ with $n \geq 1$; we claim that then $P \cap (\neg X_1) \cap \cdots \cap (\neg X_n) \neq \emptyset$. Otherwise,

$$P \subseteq X_1 \cup \cdots \cup X_n,$$

so $P$ would fork over $A$. This proves the claim.

It follows that $\Phi$ extends to a type $p \in \mathrm{St}_x(D)$ that contains all $\neg X$ with $X \in \Psi$, and thus $p$ does not fork over $A$. □

**Lemma 1.7.** *Let $M \supseteq A$ be $\kappa$-saturated where $\kappa$ is an infinite cardinal $> |A|$, and let $p \in \mathrm{St}_x(M)$. Then $p$ divides over $A$ iff $p$ forks over $A$.*

*Proof.* Suppose $p$ forks over $A$. Take $P \in p$ and definable sets $X_1, \ldots, X_n \subseteq \mathbb{U}_x$ that divide over $A$ such that $n \geq 1$ and $P \subseteq X_1 \cup \cdots \cup X_n$. Take a finite tuple $a$ in the model $M$ such that $P$ is $Aa$-definable. After applying an automorphism of $\mathbb{U}$ over $Aa$ we can arrange that all $X_i$ are defined over $M$, and then $X_i \in p$ for some $i$. □

It is even simpler to show that, given a global type $\mathbf{p} \in \mathrm{St}(\mathbb{U}_x)$, we have:

$$\mathbf{p} \text{ divides over } A \iff \mathbf{p} \text{ forks over } A.$$

**Lemma 1.8.** *Let $p \in \mathrm{St}_x(A)$. Then $p$ does not fork over $A$ iff $p$ has a global extension $\mathbf{p} \in \mathrm{St}(\mathbb{U}_x)$ that does not fork over $A$.*

*Proof.* If $p$ does not fork over $A$, then a proof like that of Lemma 1.6 yields a global extension $\mathbf{p} \in \mathrm{St}_x(\mathbb{U})$ that doesn't fork over $A$. The converse is obvious. □

Note that if a global type $\mathbf{p} \in \mathrm{St}(\mathbb{U}_x)$ is $A$-invariant, then it doesn't divide over $A$, and hence doesn't fork over $A$.

**Lemma 1.9.** *Let $X \subseteq \mathbb{U}_x$ and $Y \subseteq \mathbb{U}_y$ and $f : X \to Y$ all be $A$-definable. Let $a \in X$, $A \subseteq B$, and suppose $\mathrm{tp}_X(a|B)$ doesn't fork over $A$. Then $\mathrm{tp}_Y(f(a)|B)$ doesn't fork over $A$.*

*Proof.* Suppose towards a contradiction that $f(a) \in P \in \mathrm{Def}(Y|B)$ and $P$ forks over $A$. Then $P \subseteq P_1 \cup \cdots \cup P_n$ where $n \geq 1$ and where the definable sets $P_1, \ldots, P_n \subseteq \mathbb{U}_y$ divide over $A$. By shrinking $P$ and $P_1, \ldots, P_n$ we can arrange that $P_1, \ldots, P_n \subseteq f(X)$. Then

$$a \in f^{-1}(P) \in \mathrm{Def}(X|B), \quad f^{-1}(P) \subseteq f^{-1}(P_1) \cup \cdots \cup f^{-1}(P_n),$$

and $f^{-1}(P_1), \ldots, f^{-1}(P_n)$ divide over $A$, contradicting the assumption. □

Let $\Phi(x)$ be a finitely satisfiable set of $L(U)$-formulas in $x$, not necessarily a partial type. Then $\Phi(x)$ generates a partial type $[\Phi(x)]$ consisting of the conjunctions $\phi_1(x) \wedge \cdots \wedge \phi_n(x)$ with $\phi_1, \ldots, \phi_n \in \Phi$. We say that $\Phi(x)$ *divides over* $A$ if $[\Phi(x)]$ divides over $A$. (If $\Phi(x)$ is already a partial type, this agrees with the previous definition.) The following is an easy consequence of Lemma 1.5.

**Lemma 1.10.** *Let $\Phi(x, y)$ be a set of $L(A)$-formulas $\phi(x, y)$ and suppose $b \in \mathbb{U}_y$ is such that $\Phi(x, b)$ is finitely satisfiable. Then $\Phi(x, b)$ divides over $A$ iff there is an $A$-indiscernible sequence $(b_n)$ in $\mathbb{U}_y$ with $b_0 = b$ such that $\bigcup_n \Phi(x, b_n)$ is not finitely satisfiable.*

**Stable separation.** Let $\Phi(x, y)$ and $\Psi(x, y)$ be partial types over $A$, both consisting of formulas $\phi(x, y)$. We say that $\Phi, \Psi$ is *stably separated* if there is no $A$-indiscernible sequence $\{(a_n, b_n)\}$ in $\mathbb{U}_{x,y} = \mathbb{U}_x \times \mathbb{U}_y$ such that for all $m, n$:

$$m < n \implies \models \Phi(a_m, b_n), \qquad m > n \implies \models \Psi(a_m, b_n).$$

By Ramsey's theorem and compactness the following are equivalent:

(1) $\Phi, \Psi$ is stably separated;
(2) there is no sequence $\{(a_n, b_n)\}$ in $\mathbb{U}_{x,y}$ such that $\models \Phi(a_m, b_n)$ for all $m < n$ and $\models \Psi(a_m, b_n)$ for all $m > n$;
(3) there is $N \in \mathbb{N}^{\geq 1}$ such that there are no $(a_0, b_0), \ldots, (a_N, b_N)$ in $\mathbb{U}_{x,y}$ with $\models \Phi(a_m, b_n)$ for all $m < n$ and $\models \Psi(a_m, b_n)$ for all $m > n$;
(4) there are $N \in \mathbb{N}^{\geq 1}$, $\phi \in \Phi$, $\psi \in \Psi$ such that for all $(a_0, b_0), \ldots, (a_N, b_N)$ in $\mathbb{U}_{x,y}$, either $\models \neg\phi(a_m, b_n)$ for some $m < n$, or $\models \neg\psi(a_m, b_n)$ for some $m > n$.

The definition of "stably separated" mentions $A$, but by the equivalences above $\Phi, \Psi$ being stably separated doesn't depend on the choice of $A$ such that $\Phi$ and $\Psi$ are over $A$. If $\Phi, \Psi$ is stably separated, then clearly $\Phi \cup \Psi$ is not finitely satisfiable, and $\breve{\Phi}(y,x), \breve{\Psi}(y,x)$ is also stably separated. It follows from the equivalence with (3) that if $\Phi, \Psi$ is stably separated, so is $\Psi, \Phi$ (symmetry).

Let $\Phi(x,y), \Psi(x,y)$ be partial types over $A$ as before, and let $\mathbf{q} \in \mathrm{St}_y(\mathbb{U})$ be $A$-invariant. Note that if $p \in \mathrm{St}_x(A)$, then either all $a \models p(x)$ satisfy $\Phi(a,y) \subseteq \mathbf{q}(y)$, or all $a \models p(x)$ satisfy $\Phi(a,y) \not\subseteq \mathbf{q}(y)$. We define:

$$\Phi, \Psi \text{ is } \mathbf{q}\text{-separated}$$
$$:\Longleftrightarrow$$
$$\text{for all } (a,b) \in \mathbb{U}_{x,y}, \text{ if } \Phi(a,y) \subseteq \mathbf{q}(y),\ b \models \mathbf{q}{\restriction}A, \text{ and } p(x) \cup \Psi(x,b) \text{ is}$$
$$\text{finitely satisfiable with } p(x) := \mathrm{tp}(a|A), \text{ then } p(x) \cup \Psi(x,b) \text{ divides over } A.$$

**Lemma 1.11.** *Suppose $\Phi, \Psi$ is stably separated. Then $\Phi, \Psi$ is $\mathbf{q}$-separated (and thus by symmetry, $\Psi, \Phi$ is $\mathbf{q}$-separated).*

*Proof.* Let $(a,b) \in \mathbb{U}_{x,y}$ be such that $\Phi(a,y) \subseteq \mathbf{q}(y)$, $b \models \mathbf{q}{\restriction}A$, and the set $p(x) \cup \Psi(x,b)$ is finitely satisfiable where $p(x) := \mathrm{tp}(a|A)$. Our job is to show that $p(x) \cup \Psi(x,b)$ divides over $A$. Suppose it doesn't. Take a $\mathbf{q}$-indiscernible sequence $(b_n)$ in $\mathbb{U}_y$ with $b_0 = b$. Then by Lemma 1.10 $p(x) \cup \bigcup_n \Psi(x,b_n)$ is finitely satisfiable. By Lemma 1.4 and subsequent remark we can choose recursively elements $a_0, a_1, \dots \in \mathbb{U}_x$ and $c_0, c_1, \dots \in \mathbb{U}_y$ such that for all $n$,

(1) $a_n \models p(x) \cup \bigcup_{m<n} \Psi(x, c_m)$;
(2) $c_n \models \mathbf{q}{\restriction}Aa_0 \dots a_{n-1}c_0 \dots c_{n-1}$.

To choose $a_n$ for $n > 0$, keep in mind that the initial segment $c_0, c_1, \dots, c_{n-1}$ begins an $A$-indiscernible sequence of the same type over $A$ as $(b_i)_{i\in\mathbb{N}}$. If $m > n$, then by (1) we have $\models \Psi(a_m, c_n)$. If $m < n$, then by (1) we have $a_m \models p(x)$, so $\Phi(a_m, y) \subseteq \mathbf{q}(y)$, and hence $\models \Phi(a_m, c_n)$. Thus $\Psi, \Phi$ is not stably separated, contradicting the assumption that $\Phi, \Psi$ is stably separated. ☐

**Stable Relations.** Let $R \subseteq \mathbb{U}_x \times \mathbb{U}_y = \mathbb{U}_{x,y}$ in what follows. We say that the relation $R$ is *stable over $A$* if $R$ is $A$-invariant and for all $a, a' \in \mathbb{U}_x$, $b, b' \in \mathbb{U}_y$ with $R(a,b)$ and $\neg R(a',b')$, the pair $\mathrm{tp}((a,b)|A)$, $\mathrm{tp}((a',b')|A)$ of $(x,y)$-types over $A$ is stably separated. Note that if $R$ is stable over $A$, so are $\neg R$ and $\breve{R} \subseteq \mathbb{U}_y \times \mathbb{U}_x$, and $R$ is stable over every $B \supseteq A$. It is also clear from this definition that if $I$ is any index set and $R_i \subseteq \mathbb{U}_x \times \mathbb{U}_y$ is stable over $A$ for all $i \in I$, then $\bigcup_i R_i$ and $\bigcap_i R_i$ are stable over $A$.

**Lemma 1.12.** *Suppose $R$ is $A$-invariant. Then the following are equivalent:*

(1) *$R$ is not stable over $A$;*
(2) *there is an $A$-indiscernible sequence $\{(a_n, b_n)\}$ in $\mathbb{U}_{x,y}$ such that $R(a_m, b_n)$ for all $m < n$ and $\neg R(a_m, b_n)$ for all $m > n$.*

*Proof.* Suppose $\{(a_n, b_n)\}$ is as in (2). Put $a = a_0, b = b_1, a' = a_1, b' = b_0$. Then $R(a, b)$ and $\neg R(a', b')$, and $\{(a_n, b_n)\}$ witnesses that the pair

$$\mathrm{tp}((a, b)|A),\ \mathrm{tp}((a', b')|A)$$

is not stably separated, and so $R$ is not stable. Conversely, let $a, a' \in \mathbb{U}_x$ and $b, b' \in \mathbb{U}_y$ be such that $R(a, b)$ and $\neg R(a', b')$, and let $\{(a_n, b_n)\}$ be an $A$-indiscernible sequence in $\mathbb{U}_{x,y}$ witnessing that $\mathrm{tp}((a, b)|A)$, $\mathrm{tp}((a', b')|A)$ is not stably separated. Then this sequence is as in (2). $\square$

An immediate consequence of Lemma 1.12 is that if $X \subseteq \mathbb{U}_x$ and $Y \subseteq \mathbb{U}_y$ are $A$-invariant, then $X \times Y$ is stable over $A$.

**Lemma 1.13.** *Let $p \in \mathrm{St}_x(A)$, let $\mathbf{q} \in \mathrm{St}_y(\mathbb{U})$ be $A$-invariant, and suppose $R$ is stable over $A$. Let $a, a'$ range over $\mathbb{U}_x$ and $b, b'$ over $\mathbb{U}_y$.*

(1) *Assume $R(a, b)$, $a, a' \models p(x)$, $b \models \mathbf{q} \restriction Aa$, $b' \models \mathbf{q} \restriction A$ and $\mathrm{tp}(a'|Ab')$ does not divide over $A$. Then $R(a', b')$.*
(2) *Assume $a, a' \models p$ and $b, b' \models \mathbf{q} \restriction A$, and the types $\mathrm{tp}(a|Ab)$ and $\mathrm{tp}(a'|Ab')$ do not divide over $A$. Then $R(a, b) \iff R(a', b')$.*

*Proof.* For (1), put $\Phi(x, y) := \mathrm{tp}((a, b)|A)$, $\Psi(x, y) := \mathrm{tp}((a', b')|A)$, so

$$\Phi(a, y) = \mathrm{tp}(b|Aa), \qquad \Psi(x, b') = \mathrm{tp}(a'|Ab').$$

Suppose towards a contradiction that $\neg R(a', b')$. Then the pair $\Phi, \Psi$ is stably separated, so $\mathbf{q}$-separated by Lemma 1.11. Now $\Phi(a, y) = \mathrm{tp}(b|Aa) = \mathbf{q} \restriction Aa$, hence $\Phi(a, y) \subseteq \mathbf{q}(y)$. Also,

$$p(x) = \mathrm{tp}(a'|A) \subseteq \Psi(x, b'),$$

and hence $p(x) \cup \Psi(x, b')$ is finitely satisfied. Therefore $p(x) \cup \Psi(x, b') = \Psi(x, b') = \mathrm{tp}(a'|Ab')$ divides over $A$, contradicting the assumption in (1).

To prove (2), take $c \in \mathbb{U}_y$ such that $c \models \mathbf{q} \restriction Aa$. If $R(a, c)$, then by (1) applied to the pairs $(a, c)$ and $(a, b)$ we otain $R(a, b)$, and (1) applied to $(a, c)$ and $(a', b')$ gives $R(a', b')$. If $\neg R(a, c)$ we argue in the same way with the stable relation $\neg R$ to obtain $\neg R(a, b)$ and $\neg R(a', b')$. $\square$

**Lemma 1.14.** *Assume $R$ is stable over $M$, let $p \in \mathrm{St}_x(M)$, $q \in \mathrm{St}_y(M)$, and let $(a, b)$ range over $p(\mathbb{U}_x) \times q(\mathbb{U}_y)$. Then the following are equivalent:*

(i) *$R(a, b)$ for some $(a, b)$ such that $\mathrm{tp}(a|Mb)$ does not divide over $M$;*
(ii) *$R(a, b)$ for all $(a, b)$ such that $\mathrm{tp}(a|Mb)$ does not divide over $M$.*
(iii) *$R(a, b)$ for some $(a, b)$ such that $\mathrm{tp}(a|Mb)$ does not fork over $M$;*
(iv) *$R(a, b)$ for all $(a, b)$ such that $\mathrm{tp}(a|Mb)$ does not fork over $M$.*

*These four conditions are also equivalent to each of the four conditions obtained by replacing "$\mathrm{tp}(a|Mb)$" with "$\mathrm{tp}(b|Ma)$".*

*Proof.* By Lemma 1.1 we have a global $M$-invariant $\mathbf{q} \in \mathrm{St}_y(\mathbb{U})$ that extends $q$. Using $\mathbf{q}$ the implication (i) $\Rightarrow$ (ii) follows from (2) of Lemma 1.13.

For the converse it is enough to produce $(a, b)$ such that $\mathrm{tp}(a|Ab)$ does not divide over $M$. Take a global $M$-invariant $\mathbf{p} \in \mathrm{St}_x(\mathbb{U})$ that extends $p$, so $\mathbf{p}$ doesn't divide over $M$. Then for any $b$, $\mathbf{p}\restriction Mb$ doesn't divide over $M$, so if $a$ realizes $\mathbf{p}\restriction Mb$, then $\operatorname{tp}(a|Ab)$ doesn't divide over $M$.

This construction of a pair $(a,b)$ works also with "fork" instead of "divide" and thus produces a pair $(a,b)$ such that $\operatorname{tp}(a|Mb)$ doesn't fork over $M$, and hence also doesn't divide over $M$. This yields (ii) $\Rightarrow$ (iii), and (iv)$\Rightarrow$ (iii). The direction (iii) $\Rightarrow$ (i) follows in view of "not forking" implying "not dividing". The latter also gives (iv) $\Rightarrow$ (ii). This proves the equivalence of conditions (i)–(iv).

By symmetry considerations, the four conditions obtained from these by replacing "$\operatorname{tp}(a|Mb)$" with "$\operatorname{tp}(b|Ma)$" are also pairwise equivalent; to show they are equivalent to (i)–(iv) it is enough to have a pair $(a,b)$ such that $\operatorname{tp}(a|Mb)$ and $\operatorname{tp}(b|Ma)$ don't divide over $M$. To get such a pair we use Lemma 1.1 to obtain an $M$-invariant $\mathbf{r}\in \operatorname{St}_{x,y}(\mathbb{U})$ that extends $p\cup q$. Then $\mathbf{r}$ doesn't divide over $M$, so any $(a,b)\models \mathbf{r}\restriction M$ has the desired property. □

Let $\mathbf{q}\in \operatorname{St}_y(\mathbb{U})$, and $a\in \mathbb{U}_x$. Then

$$(\mathbf{q}\restriction Aa)(\mathbb{U}_y) \;=\; \{b\in \mathbb{U}_y:\; b\models \mathbf{q}\restriction Aa\} \;\neq \emptyset.$$

Any two elements of $(\mathbf{q}\restriction Aa)(\mathbb{U})$ are $Aa$-conjugate, so for $A$-invariant $R$,

$$(\mathbf{q}\restriction Aa)(\mathbb{U}_y)\subseteq R(a) \iff (\mathbf{q}\restriction Aa)(\mathbb{U}_y)\cap R(a)\neq\emptyset.$$

It follows that if $R$ is $A$-invariant and $B\supseteq A$, then

$$(\mathbf{q}\restriction Aa)(\mathbb{U}_y)\subseteq R(a) \iff (\mathbf{q}\restriction Ba)(\mathbb{U}_y)\subseteq R(a).$$

In the rest of this subsection the multivariables $x,y$ are similar.

**Lemma 1.15.** *Suppose $R$ is stable over $M$, $\mathbf{q},\mathbf{r}\in \operatorname{St}_y(\mathbb{U})$ do not divide over $M$, and $\mathbf{q}\restriction M=\mathbf{r}\restriction M$. Then for all $a\in\mathbb{U}_x=\mathbb{U}_y$,*

$$(\mathbf{q}\restriction Ma)(\mathbb{U}_y)\subseteq R(a) \iff (\mathbf{r}\restriction Ma)(\mathbb{U}_y)\subseteq R(a).$$

*Proof.* Let $a\in\mathbb{U}_x$ and suppose $(\mathbf{q}\restriction Ma)(\mathbb{U}_y)\subseteq R(a)$, that is, $R(a,b)$ for all $b\models \mathbf{q}\restriction Ma$. Put $p:=\operatorname{tp}_x(a|M)$, take an $M$-invariant $\mathbf{p}\in\operatorname{St}_x(\mathbb{U})$ that extends $p$. Take $c\models\mathbf{q}\restriction Ma$, so $R(a,c)$ and $\operatorname{tp}(c|Ma)$ doesn't divide over $M$. Then Lemma 1.14 with $q:=\mathbf{q}\restriction M$ yields that $R(a',c)$ for all $a'\models\mathbf{p}\restriction Mc$. Taking such an $a'$ and noting that $c\models\mathbf{r}\restriction M=\mathbf{q}\restriction M$ we can apply the same lemma again to conclude that $R(a,b)$ for all $b\models\mathbf{r}\restriction Ma$, that is, $(\mathbf{r}\restriction Ma)(\mathbb{U}_y)\subseteq R(a)$. □

Let $S=S^{\mathrm{nd}}_{A,y}$ be the set of global types $\mathbf{q}\in\operatorname{St}_y(\mathbb{U})$ that do not divide over $A$. We define an equivalence relation $E$ on $S$ by: $\mathbf{q}\;E\;\mathbf{r}$ if and only if

$$\text{for every } R\subseteq\mathbb{U}_x\times\mathbb{U}_y \text{ that is stable over } A\text{, and all } a\in\mathbb{U}_y,$$
$$(\mathbf{q}\restriction Aa)(\mathbb{U}_y)\subseteq R(a) \iff (\mathbf{r}\restriction Aa)(\mathbb{U}_y)\subseteq R(a).$$

**Corollary 1.16.** *Suppose $|A|,|y|\le|L|$. Then $|S/E|\le 2^{|L|}$.*

*Proof.* Take $M\supseteq A$ with $|M|\le|L|$. By Lemma 1.15 and the remarks preceding this lemma, we have for all $\mathbf{q},\mathbf{r}\in S$, if $\mathbf{q}\restriction M=\mathbf{r}\restriction M$, then $\mathbf{q}\;E\;\mathbf{r}$. This yields the desired estimate since $|\operatorname{Def}_x(\mathbb{U}|M)|\le|L|$. □

**The $A$-topology.** The $A$-definable subsets of $\mathbb{U}_x$ form a basis for a certain topology on $\mathbb{U}_x$, the $A$-topology. The $A$-open subsets of $\mathbb{U}_x$ are the unions of $A$-definable subsets of $\mathbb{U}_x$; instead of *$A$-open* one also uses the term $\bigvee$-*definable over $A$*. The $A$-closed subsets of $\mathbb{U}_x$ are the intersections of $A$-definable subsets of $\mathbb{U}_x$, equivalently, the sets $\Phi(\mathbb{U}_x) \subseteq \mathbb{U}_x$, with $\Phi(x)$ a set of $L(A)$-formulas in $x$; instead of *$A$-closed* one also uses the terms $\bigwedge$-*definable over $A$* and *type-definable over $A$*. The $A$-definable subsets of $\mathbb{U}_x$ are exactly the $A$-clopen subsets of $\mathbb{U}_x$, and $\mathbb{U}_x$ is compact (but usually not hausdorff) in the $A$-topology. For related basic facts on type-definability, see [2]. One use of $A$-closed sets (respectively, $A$-open sets) is the possibility of forming inverse limits (respectively, direct limits); this is also a reason for allowing $x$ to be an infinite (but small) tuple of variables. A related viewpoint is that $A$-open sets are like locally compact spaces, with $A$-definable and $A$-closed sets more like compact spaces. Note that $A$-open and $A$-closed subsets of $\mathbb{U}_x$ are $A$-invariant.

For use in the next section we prove here a general fact, Lemma 1.17, which is implicit in Hrushovski's proof of his stabilizer theorem 3.4 in [3]. First some introductory remarks.

Let $X \subseteq \mathbb{U}_x$ be $A$-closed, and let $(X_i)_{i\in I}$ be a defining system for $X$ over $A$ in the sense of [2], that is, $I$ is small, $X_i \in \mathrm{Def}(\mathbb{U}_x|A)$ for all $i \in I$, and $X = \bigcap_i X_i$. Then for any small collection $\Phi(x)$ of formulas $\phi(x)$ over $A$,

$$\Phi(X) = \bigcap_i \Phi(X_i) = \bigcap_{i\in I,\ \phi\in\Phi} \phi(X_i),$$

so $\Phi(X)$ is $A$-closed with defining system $(\phi(X_i))_{i\in I,\ \phi\in\Phi}$.

Assume also that $p \in \mathrm{St}_x(A)$. Then $p(X) = \emptyset$ or $p(X) = p(\mathbb{U}_x)$. This is because $p(X) \subseteq p(\mathbb{U}_x)$, $p(X)$ is an $A$-invariant subset of $\mathbb{U}_x$, and $p(\mathbb{U}_x)$, as an orbit, is a minimal nonempty $A$-invariant subset of $\mathbb{U}_x$.

**Lemma 1.17.** *Let $X \subseteq \mathbb{U}_x$ be $M$-closed and $E$ an $M$-closed equivalence relation on $X$ such that $X/E$ is small. Let $p \in \mathrm{St}_x(M)$ and $p(X) \neq \emptyset$. Then $p(X) \subseteq C$ for a (necessarily unique) $E$-class $C \in X/E$, and this class $C$ is $M$-closed.*

*Proof.* Let $(X_i), (E_i)$ be a directed defining system for $X, E$ as defined in [2]. Note that $p(X) = p(\mathbb{M}_x)$ by the remarks above. We extend $p$ to an $M$-invariant global type $\mathbf{p} \in \mathrm{St}(\mathbb{U}_x)$, and then take an elementary extension $\mathbb{U}'$ of $\mathbb{U}$ where $\mathbf{p}$ is realized by an element $a \in \mathbb{U}'_x$. This yields the definable sets $X'_i \subseteq \mathbb{U}'_x$ and $E'_i \subseteq X'_i \times X'_i$, the set $X' = \bigcap_i X'_i \subseteq \mathbb{U}'_x$ and the equivalence relation $E' = \bigcap_i E'_i$ on $X'$. If $C \in X/E$, then $C' \in X'/E'$, and the map

$$C \mapsto C' \ :\ X/E \to X'/E'$$

is a bijection (Lemma 3.4 of [2]). We have $a \in \mathbf{p}(\mathbb{U}'_x) \subseteq p(\mathbb{U}'_x)$. We claim that $p(\mathbb{U}'_x) \subseteq X'$: to see this, note that for each $i \in I$ we have $p(\mathbb{U}_x) = p(X) \subseteq X_i$, so we get $\phi(x) \in p(x)$ with $\phi(\mathbb{U}_x) \subseteq X_i$, hence $p(\mathbb{U}'_x) \subseteq \phi(\mathbb{U}'_x) \subseteq X'_i$. This yields the claim. It follows that we have $C \in X/E$ such that $a \in C'$. Take

$c \in C$. For $i \in I$ we have $a \in C' \subseteq E_i(c)'$, so $E_i(c) \in \mathbf{p}$. On the other hand, if $d \in X$ and $(c,d) \notin E$, then there is $i$ such that $E_i(c) \cap E_i(d) = \emptyset$ and thus $E_i(d) \notin \mathbf{p}$. In this way $\mathbf{p}$ picks out a unique $E$-class, namely $C$. But $\mathbf{p}$ is $M$-invariant, so $C$ is $M$-invariant, and since $C$ is type-definable (over $Mc$), it follows that $C$ is $M$-closed.

Next we claim that $p(\mathbb{U}_x) \cap E_i(c) \neq \emptyset$ for all $i$. To see this, suppose $i \in I$ is such that $p(\mathbb{U}_x) \cap E_i(c) = \emptyset$. Then we have $\phi(x) \in p(x)$ such that $\phi(\mathbb{U}_x) \cap E_i(c) = \emptyset$, and since $\phi(\mathbb{U}_x) \in \mathbf{p}$, this would give $E_i(c) \notin \mathbf{p}$, a contradiction. This proves our claim, and since $C = \bigcap_i E_i(c)$, compactness yields $p(\mathbb{U}_x) \cap C \neq \emptyset$. Since $p(\mathbb{U}_x)$ is a minimal nonempty $M$-invariant subset of $\mathbb{U}_x$, this yields $p(X) = p(\mathbb{U}_x) \subseteq C$. □

It will also be useful to extend some earlier constructions to $A$-open sets.

Let $X \subseteq \mathbb{U}_x$ be $A$-open. Then we define:

$$\begin{aligned} \mathrm{Def}(X) &:= \{P \subseteq X : \ P \text{ is definable}\}, \\ \mathrm{Def}(X|A) &:= \{P \subseteq X : \ P \text{ is } A\text{-definable}\}, \end{aligned}$$

and for any $a \in X$ and any parameter set $D$ (not necessarily small),

$$\begin{aligned} \mathrm{tp}_X(a|D) &:= \{P \in \mathrm{Def}(X) : \ a \in P\}, \\ \mathrm{St}(X|D) &:= \{\mathrm{tp}_X(a|D) : \ a \in X\}, \end{aligned}$$

in particular, $\mathrm{tp}_X(a|A) := \{P \in \mathrm{Def}(X|A) : \ a \in P\}$. By an $X$-formula we mean an $L$-formula $\phi(x,y)$ such that $\phi(\mathbb{U}_x, b) \subseteq X$ for all $b \in \mathbb{U}_y$ (and thus $\phi(\mathbb{U}_x, b) = \phi(X, b)$ for all $b \in \mathbb{U}_y$). Because $X$ is $A$-open, every $P \in \mathrm{Def}(X)$ equals $\phi(\mathbb{U}_x, b)$ for some $X$-formula $\phi(x,y)$ and some $b \in \mathbb{U}_y$.

**Keisler measures.** Let $X \subseteq \mathbb{U}_x$ be $A$-open. A *Keisler measure* on $X$ is a finitely additive measure

$$\mu : \mathrm{Def}(X) \to [0,\infty] = \mathbb{R}^{\geq 0} \cup \{\infty\}, \qquad (\text{in particular, } \mu(\emptyset) = 0)^2.$$

Let $\mu : \mathrm{Def}(X) \to [0,\infty]$ be a Keisler measure. Then we have for each $X$-formula $\phi(x,y)$ the function

$$\mu_\phi : M_y \to [0,\infty], \quad \mu_\phi(b) = \mu\phi(X,b).$$

We say that $\mu$ is *$A$-invariant* if $\mu(P) = \mu(\sigma P)$ for all $P \in \mathrm{Def}(X)$ and $\sigma \in \mathrm{Aut}(\mathbb{U}|A)$, equivalently, for each $X$-formula $\phi(x,y)$ we have $\mu_\phi(b) = \mu_\phi(c)$ whenever $b, c \in \mathbb{U}_y$ are $A$-conjugate (and so $\mu_\phi$ induces a function $\mu_\phi : \mathrm{St}_y(A) \to [0,1]$ by $\mu_\phi(\mathrm{tp}(b|A)) := \mu_\phi(b)$ for $b \in \mathbb{U}_y$). We say that $\mu$ is $A$-*definable* (in $\mathbb{U}$) if $\mu$ is $A$-invariant and each function $\mu_\phi : \mathbb{U}_y \to [0,1]$ as above is $A$-continuous, equivalently, each induced function $\mu_\phi : \mathrm{St}_y(A) \to [0,1]$ is continuous.

[2] For $A$-definable $X$ one can also impose $\mu(X) = 1$.

**Ideals.** Let $\mathcal{C}$ be a collection of subsets of $\mathbb{U}_x$. We say that $\mathcal{C}$ is *A-invariant* if $\sigma P \in \mathcal{C}$ for all $P \in \mathcal{C}$ and $\sigma \in \mathrm{Aut}(\mathbb{U}|A)$. Note that then for every $L$-formula $\phi(x,y)$ we have a unique set $E_{\mathcal{C},\phi} \subseteq \mathrm{St}_y(A)$ of types such that for all $b \in \mathbb{U}_y$,

$$\phi(\mathbb{U}_x, b) \in \mathcal{C} \iff \mathrm{tp}(b|A) \in E_{\mathcal{C},\phi},$$

and that this equivalence determines $\mathcal{C}$ in terms of the sets $E_{\mathcal{C},\phi}$. The collections $\mathcal{C}$ we have in mind are ideals in $\mathrm{Def}(X)$ for $A$-open $X$.

*In the rest of this subsection $X \subseteq \mathbb{U}_x$ is A-open and $I$ is an ideal of* $\mathrm{Def}(X)$, so $I$ is a collection of definable subsets of $X$ such that $\emptyset \in I$, and for all $P, Q \in \mathrm{Def}(X)$,

$$P, Q \in I \implies P \cup Q \in I, \qquad P \subseteq Q \in I \implies P \in I.$$

We say that $I$ is *proper* if $I \neq \mathrm{Def}(X)$. (This notion is relative to the ambient $X$: if $Y \subseteq X$ is $A$-open, then $\mathrm{Def}(Y)$ is an improper ideal of $\mathrm{Def}(Y)$ but can be proper as an ideal of $\mathrm{Def}(X)$.)

*Examples.* $\{P \in \mathrm{Def}(X) : \ P \text{ forks over } A\} \cup \{\emptyset\}$, is an $A$-invariant ideal of $\mathrm{Def}(X)$, the *forking ideal over A* (in $X$). If $\mu$ is an $A$-invariant Keisler measure on $X$, then

$$\{P \in \mathrm{Def}(X) : \mu(P) = 0\}$$

is an $A$-invariant ideal of $\mathrm{Def}(X)$, called the *zero ideal of* $\mu$, and is a proper ideal if $\mu(P) > 0$ for some $P \in \mathrm{Def}(X)$.

We say that $I$ is *A-definable* (respectively, *A-closed, A-open*) if for each $L$-formula $\phi(x,y)$ the set $\{b \in \mathbb{U}_y : \ \phi(\mathbb{U}_x, b) \in I\}$ is $A$-definable (respectively, $A$-closed, $A$-open). Note that if $I$ is $A$-closed or $A$-open, then $I$ is $A$-invariant. If $\mu$ is an $A$-definable Keisler measure on $X$, then its zero ideal is clearly $A$-closed.

**Definition.** *I is S1* over $A$ if $I$ is $A$-invariant, and for every $L$-formula $\phi(x,y)$ and $A$-indiscernible sequence $(b_n)$ in $\mathbb{U}_y$, if $\phi(\mathbb{U}_x, b_0) \in \mathrm{Def}(X)$ (and thus $\phi(\mathbb{U}_x, b_n) \in \mathrm{Def}(X)$ for all $n$), and $\phi(\mathbb{U}_x, b_m) \cap \phi(\mathbb{U}_x, b_n) \in I$ for all $m \neq n$, then $\phi(\mathbb{U}_x, b_n) \in I$ for some $n$ (and hence for all $n$).

For $A$-invariant $I$, the following are equivalent:

(1) $I$ is S1 over $A$;
(2) for every $A$-definable relation $R \subseteq X \times \mathbb{U}_y$ and every sequence $(b_n)$ in $\mathbb{U}_y$ with $\breve{R}(b_n) \notin I$ for all $n$, there are $m < n$ with $\breve{R}(b_m) \cap \breve{R}(b_n) \notin I$.

This equivalence follows as usual by Ramsey's theorem and saturation.

The zero ideal of an $A$-invariant Keisler measure $\mu$ on $X$ with $\mu(X) < \infty$ is clearly S1. The forking ideal over $A$ in $X$ is contained in every S1-ideal over $A$:

**Lemma 1.18.** *Suppose $I$ is S1 over $A$ and $P \in \mathrm{Def}(X)$ forks over $A$. Then $P \in I$.*

*Proof.* We can reduce to the case that $P$ divides over $A$. Suppose towards a contradiction that $P \notin I$. Take an $L$-formula $\phi(x,y)$ and $b \in \mathbb{U}_y$ such that $P = \phi(\mathbb{U}_x, b)$. Then by Lemma 1.5 we have an $A$-indiscernible sequence $(b_n)$ in $\mathbb{U}_y$ with $b_0 = b$ and $\phi(\mathbb{U}_x, b_0) \cap \cdots \cap \phi(\mathbb{U}_x, b_n) = \emptyset$ for some $n \geq 1$. Note that $\phi(\mathbb{U}_x, b_n) \in \mathrm{Def}(X)$ for all $n$. Take $m$ maximal such that $\phi(\mathbb{U}_x, b_0) \cap \cdots \cap \phi(\mathbb{U}_x, b_m) \notin I$, and put

$$P_i := \phi(\mathbb{U}_x, b_0) \cap \cdots \cap \phi(\mathbb{U}_x, b_{m-1}) \cap \phi(\mathbb{U}_x, b_{m+i}), \quad i \in \mathbb{N}.$$

Then $P_i \notin I$ for all $i$ and $P_i \cap P_j \in I$ for all $i \neq j$. The sequence

$$\{(b_0, \ldots, b_{m-1}, b_{m+i})\}_{i \in \mathbb{N}}$$

is $A$-indiscernible and $I$ is S1 over $A$, a contradiction. □

**Lemma 1.19.** *Let $X \subseteq \mathbb{U}_x$ and $Y \subseteq \mathbb{U}_y$ be $A$-definable, let $Z \subseteq \mathbb{U}_z$ be $A$-open and $J$ an ideal of* $\mathrm{Def}(Z)$ *that is S1 over $A$. Let $P \subseteq X \times Z$ and $Q \subseteq Y \times Z$ be $A$-definable, and define $R \subseteq X \times Y \subseteq \mathbb{U}_x \times \mathbb{U}_y$ by*

$$R(a,b) \iff P(a) \cap Q(b) \in J.$$

*Then $R$ is stable over $A$.*

*Proof.* It is clear that $R$ is $A$-invariant. Let $\{(a_n, b_n)\}$ be an $A$-indiscernible sequence in $\mathbb{U}_{x,y}$ with $R(a_m, b_n)$ for all $m < n$, so $a_n \in X,\ b_n \in Y$ for all $n$. By Lemma 1.12 it suffices to show that then $R(a_m, b_n)$ for some $m > n$. First, $R(a_n, b_n)$ for all $n$: otherwise, $\neg R(a_n, b_n)$ for all $n$, and so for $C_n := P(a_n) \cap Q(b_n)$ we have $C_n \notin J$ for all $n$ and $C_m \cap C_n \in J$ for all $m \neq n$, which contradicts that $J$ is S1 over $A$. Next, the sequence $\{(a_{2n}, b_{2n}, a_{2n+1}, b_{2n+1})\}$ is also $A$-indiscernible, and so is $\{(c_n, d_n)\} := \{(a_{2n+1}, b_{2n})\}$. Since $R(c_m, d_n)$ for all $m < n$, the above gives $R(c_n, d_n)$ for all $n$, that is, $R(a_{2n+1}, b_{2n})$ for all $n$. □

A set $P \subseteq X$ is said to be $I$-wide if $P \not\subseteq Y$ for all $Y \in I$. If $P \in \mathrm{Def}(X)$, then $P$ is $I$-wide iff $P \notin I$. A partial type $\Phi(x)$ is said to be *in $X$* if $\phi(\mathbb{U}_x) \subseteq X$ for all $\phi \in \Phi$, and is said to be *$I$-wide* if $\Phi$ is in $X$ and $\phi(\mathbb{U}_x)$ is $I$-wide for all $\phi \in \Phi(x)$. If $\Phi(x)$ is a partial type in $X$ over some $B$, then $\Phi$ is $I$-wide iff $\Phi(X)$ is $I$-wide.

**Lemma 1.20.** *Let $P \in \mathrm{Def}(X)$ and suppose the nonempty partial type $\Phi(x)$ is in $P$ and is $I$-wide. Then $\Phi$ extends to an $I$-wide global type $\mathbf{p} \in \mathrm{St}(P)$.*

*Proof.* If $Y \in I, Y \subseteq P$ and $\phi \in \Phi$, then $(P \setminus Y) \cap \phi(\mathbb{U}_x) \neq \emptyset$, since otherwise $\phi(\mathbb{U}_x) \subseteq Y$. This gives $\mathbf{p} \in \mathrm{St}(P)$ extending $\Phi$ such that $P \setminus Y \in \mathbf{p}$ for all $Y \in I$ with $Y \subseteq P$; such $\mathbf{p}$ is $I$-wide. □

**Theorem 1.21.** *Let $Z \subseteq \mathbb{U}_z$ be $A$-open and $J$ an ideal of* $\mathrm{Def}(Z)$ *that is S1 over $A$. Let $a \in \mathbb{U}_x, b, b' \in \mathbb{U}_y, c \in Z$ be such that $\mathrm{tp}_Z(c|Aab)$ is $J$-wide, $\mathrm{tp}(b|A) = \mathrm{tp}(b'|A)$, $\mathrm{tp}(b|Aa)$ and $\mathrm{tp}(b'|Aa)$ do not divide over $A$, and $\mathrm{tp}(a|A)$ extends to an $A$-invariant global type. Then there exists $c' \in Z$ such that $\mathrm{tp}_Z(c'|Aab')$ is $J$-wide, and*

$$\mathrm{tp}((a,c')|A) = \mathrm{tp}((a,c)|A), \qquad \mathrm{tp}((b',c')|A) = \mathrm{tp}((b,c)|A).$$

*Proof.* Let $P \subseteq \mathbb{U}_x \times Z$ and $Q \subseteq \mathbb{U}_y \times Z$ be $A$-definable with $P(a,c)$ and $Q(b,c)$. By compactness it is enough to find for any such $P, Q$ an element $c' \in Z$ such that $P(a,c')$, $Q(b',c')$ and $\operatorname{tp}_Z(c'|Aab')$ is $J$-wide, that is, it suffices to show that $P(a) \cap Q(b') \notin J$.

Let $R \subseteq \mathbb{U}_x \times \mathbb{U}_y$ be defined by $R(d,e) \Longleftrightarrow P(d) \cap Q(e) \in J$. Then $R$ is stable over $A$ by Lemma 1.19. Also $P(a) \cap Q(b) \in \operatorname{tp}_Z(c|Aab)$ and $\operatorname{tp}_Z(c|Aab)$ is $J$-wide, so $\neg R(a,b)$. Hence by part (2) of Lemma 1.13 (interchanging the roles of $x, y$), we obtain $\neg R(a,b')$, that is, $P(a) \cap Q(b') \notin J$, as desired. □

We conclude this section with three lemmas that give the existence of useful global types relative to $I$, in three slightly different situations.

**Lemma 1.22.** *Suppose $A = M$ and $I$ is proper and $M$-open. Then there is a global type $\mathbf{p} \in \operatorname{St}(X)$, finitely satisfiable in $M$, such that for all $a, b \in X$, if $a \models \mathbf{p}\restriction M$ and $b \models \mathbf{p}\restriction Ma$, then $\operatorname{tp}_X(a|Mb)$ is $I$-wide.*

*Proof.* Since $I$ is proper, we have $P \in \operatorname{Def}(X)$ such that $P \setminus Y \neq \emptyset$ for all $Y \in I$. This yields a type $p \in \operatorname{St}(X|M)$ such that $P \setminus Y \in p$ for all $M$-definable $Y \in I$. Then Lemma 1.1 yields a global type $\mathbf{p} \in \operatorname{St}(X)$ that extends $p$ and is finitely satisfiable in $M$. Let $a, b \in X$ with $a \models p$ and $b \models \mathbf{p} \restriction Ma$, and suppose towards a contradiction that $\operatorname{tp}_X(a|Mb)(X) \subseteq Y \in I$. Then by compactness we have an $L(A)$-formula $\phi(x,y)$ with $x$ and $y$ similar, such that $\models \phi(a,b)$ and $\phi(X,b) \subseteq Y$, and thus $\phi(X,b) \in I$. Now $I$ is $M$-open, so we have an $L(M)$-formula $\theta(y)$ such that $\models \theta(b)$, and $\phi(X,b') \in I$ for all $b' \in \theta(X)$. Since $\operatorname{tp}_X(b|Ma)$ is finitely satisfiable in $M$ we get $b' \in \theta(X) \cap M_y$ such that $\models \phi(a,b')$. Then $\phi(X,b')$ is an $M$-definable set in $I$, so $X \setminus \phi(X,b') \in p$ and thus $\models \neg\phi(a,b')$, a contradiction. □

The next lemma has some Fubini-type assumptions:

**Lemma 1.23.** *Suppose $A = M$, $L$ and $M$ are countable, and $I$ is $M$-closed and proper. With $y$ similar to $x$, assume also that $J$ is an ideal of $\operatorname{Def}(X^2|M)$ such that for all $M$-definable $P \subseteq X$ and $R \subseteq X^2$,*

(1) *if $P^2 \in J$, then $P \in I$;*
(2) *if $R(a) \in I$ for all $a \in X$ with $I$-wide $\operatorname{tp}(a|M)$, then $R, \breve{R} \in J$.*

*Then there exists a global type $\mathbf{p} \in \operatorname{St}(X)$, finitely satisfiable in $M$, such that $\operatorname{tp}_X(a|Mb)$ and $\operatorname{tp}_X(b|Ma)$ are $I$-wide for all $a \models \mathbf{p}\restriction M$ and $b \models \mathbf{p}\restriction Ma$.*

*Proof.* We first construct a certain $I$-wide $p \in \operatorname{St}(X|M)$, and then extend it to a global type $\mathbf{p}$ as required.

**Claim 1.** Let $P \subseteq X$ and $R_1, R_2, R_3 \subseteq X^2$ be $M$-definable such that $P \notin I$ and $P^2 \subseteq R_1 \cup R_2 \cup R_3$. Then there is an $M$-definable $Q \subseteq P$ with $Q \notin I$ such that for all $a, b \in Q$,

$$(*) \qquad R_1(a) \notin I \ \text{ or } \ \breve{R}_2(b) \notin I \text{ or } \ \breve{R}_3(c) \supseteq Q \text{ for some } c \in X(M).$$

*Proof of Claim 1.* Suppose $P \cap \breve{R}_3(c) \notin I$ for some $c \in X(M)$. Take such $c$, put $Q := P \cap \breve{R}_3(c)$, and note that then the third option in (*) holds for all $a, b \in Q$. So it remains to consider the case that $P \cap \breve{R}_3(c) \in I$ for all $c \in X(M)$. Now $I$ is $M$-closed, so $P \cap \breve{R}_3(c) \in I$ for all $c \in X$. Hence $(P \times X) \cap R_3 \in J$, by (2).

If there is an $M$-definable $Q \subseteq P$ with $Q \notin I$ such that $R_1(a) \notin I$ for all $a \in Q$, then the first option of (*) holds. So we can assume that for all $M$-definable $Q \subseteq P$ with $Q \notin I$ there is $a \in Q$ with $R_1(a) \in I$. Now $I$ is $M$-closed, so for all $a \in P$, if $\mathrm{tp}(a|M)$ is $I$-wide, then $R_1(a) \in I$. Hence by (2) again we get $(P \times X) \cap R_1 \in J$.

If there is an $M$-definable $Q \subseteq P$ with $Q \notin I$ such that $\breve{R}_2(b) \notin I$ for all $b \in Q$, then the second option of (*) holds. We now show that there are no other possibilities: Assuming there is no such $Q$ we obtain $(X \times P) \cap R_2 \in J$ as before. The three sets we showed to be in $J$ yield $R_1 \cup R_2 \cup R_3 \in J$. Then from $P^2 \subseteq R_1 \cup R_2 \cup R_3$ we get $P^2 \in J$, so $P \in I$ by (1), a contradiction. This finishes the proof of Claim 1.

**Claim 2.** There is an $I$-wide $p \in \mathrm{St}(X|M)$ such that for all $M$-definable $R_1, R_2, R_3 \subseteq X^2$, if $p(X)^2 \subseteq R_1 \cup R_2 \cup R_3$, then there exists $Q \in p$ such that (*) holds for all $a, b \in Q$.

To prove this we use that $L$ and $M$ are countable. Take an enumeration $(X_n)$ of $\mathrm{Def}(X|M)$ and an enumeration $(R_{n1}, R_{n2}, R_{n3})$ of $\mathrm{Def}(X^2|M)^3$ in which every $(R_1, R_2, R_3) \in \mathrm{Def}(X^2|M)^3$ occurs infinitely often. Choose recursively a descending sequence $P_0 \supseteq P_1 \supseteq P_2 \cdots$ in $\mathrm{Def}(X|M) \setminus I$ such that

- (a) $P_{2n} \subseteq X_n$ or $P_{2n} \subseteq \neg X_n$;
- (b) if $P_{2n}^2 \subseteq R_{n1} \cup R_{n2} \cup R_{n3}$, take $Q$ as in Claim 1 for $P_{2n}$ and $R_{n1}, R_{n2}, R_{n3}$ in the role of $P, R_1, R_2, R_3$, and set $P_{2n+1} := Q$.

It is easy to check that $p = \{P \in \mathrm{Def}(X|M) : \ P \supseteq P_n \text{ for some } n\}$ satisfies Claim 2.

Let $p$ be as in Claim 2, and take $a \in p(X)$. Let $\Gamma(a)$ be the collection of definable subsets of $X$ consisting of the sets in $p$ together with the following sets for all $M$-definable $R \subseteq X^2$:

- (i) $\neg R(a)$ if $R(a) \in I$;
- (ii) $\neg R(a)$ if $\breve{R}(a) \in I$;
- (iii) $\neg R(a)$ if $\breve{R}(c) \notin p$ for all $c \in X(M)$.

Note: $\breve{R}(c) \notin p$ for all $c \in X(M)$ if and only if $X(M) \subseteq \neg R(a)$.

**Claim 3.** Let $X_1, \ldots, X_n \in \Gamma(a)$. Then $X_1 \cap \cdots \cap X_n \neq \emptyset$.

To prove this, take $P \in p$ and $M$-definable $R_1, R_2, R_3 \subseteq X^2$ such that

$$X_1 \cap \cdots \cap X_n \supseteq P \cap \neg R_1(a) \cap \neg R_2(a) \cap \neg R_3(a),$$

with $R_1(a), \breve{R}_2(a) \in I$, and $\breve{R}_3(c) \notin p$ for all $c \in X(M)$. Suppose towards a contradiction that $X_1 \cap \cdots \cap X_n = \emptyset$. Then

$$P \subseteq R_1(a) \cup R_2(a) \cup R_3(a),$$

so $p(X) \times P \subseteq R_1 \cup R_2 \cup R_3$ and thus $p(X)^2 \subseteq R_1 \cup R_2 \cup R_3$. Then by Claim 2 we have $R_1(a) \notin I$ or $\breve{R}_2(a) \notin I$ or $\breve{R}_3(c) \in p$ for some $c \in X(M)$, a contradiction. This proves Claim 3.

**Claim 4.** Let $X_1, \dots, X_n \in \Gamma(a)$. Then $X_1 \cap \dots \cap X_n \cap X(M) \neq \emptyset$.

Suppose towards a contradiction that $X_1 \cap \dots \cap X_n \cap X(M) = \emptyset$. As in the proof of Claim 3 we get $P \in p$ and $M$-definable $R_1, R_2, R_3 \subseteq X^2$ such that $R_1(a), \breve{R}_2(a) \in I$, and $\breve{R}_3(c) \notin p$ for all $c \in X(M)$, and

$$P(M) \subseteq R_1(a) \cup R_2(a) \cup R_3(a),$$

that is, $X(M) \subseteq \neg R(a)$ for $R \subseteq X^2$ defined by

$$R(b, c) \iff P(c) \text{ and } \neg R_1(b, c) \text{ and } \neg R_2(b, c) \text{ and } \neg R_3(b, c).$$

Thus by (iii) we have $\neg R(a) \in \Gamma(a)$, that is

$$\neg P \cup R_1(a) \cup R_2(a) \cup R_3(a) \in \Gamma(a),$$

contradicting $P, \neg R_1(a), \neg R_2(a), \neg R_3(a) \in \Gamma(a)$ in view of Claim 3.

Thus $\Gamma(a)$ extends by Claim 4 and Lemma 1.1 to a global type $\mathbf{p} \in \mathrm{St}(X)$ that is finitely satisfiable in $M$. Take $b \models \mathbf{p} \restriction Ma$. It is easy to check that $\mathrm{tp}_X(b|Ma)$ is $I$-wide because of (i).

It remains to show that $\mathrm{tp}_X(a|Mb)$ is $I$-wide. Let $P \in \mathrm{tp}_X(a|Mb)$. Then $a \in P = R(b)$ where $R \subseteq X^2$ is $M$-definable. Towards a contradiction, suppose that $P \in I$, that is, $R(b) \in I$. Now $a$ and $b$ both realize $p = \mathbf{p} \restriction M$, so they are $M$-conjugate, and thus $R(a) \in I$. Hence $\neg\breve{R}(a) \in \mathbf{p} \restriction Ma$ by (ii), so $b \in \neg\breve{R}(a)$, contradicting $a \in R(b)$. □

**Lemma 1.24.** *Suppose $L$ and $A$ are countable, and $I$ is proper and $A$-invariant. Then there is a countable $M \supseteq A$ and a global type $\mathbf{p} \in \mathrm{St}(X)$, finitely satisfiable in $M$, such that for all $a, b \in X$, if $a \models \mathbf{p} \restriction M$, $b \models \mathbf{p} \restriction Ma$, then $\mathrm{tp}_X(a|Mb)$ is $I$-wide.*

*Proof.* Assume that $\mathrm{Th}(\mathbb{U})$ has definable Skolem functions and $\kappa(\mathbb{U}) > \beth_{\omega_1}$. (After proving the lemma for this case we shall reduce the general case to this special case.) For any $B$, let $\langle B \rangle$ be the substructure of $\mathbb{U}$ generated by $B$; by definability of Skolem functions, this is a (small) elementary submodel of $\mathbb{U}$. Since $I$ is proper, there is for each $M$ an $I$-wide $p \in \mathrm{St}(X|M)$. Thus by transfinite recursion we obtain a sequence $(a_i : i < \beth_{\omega_1})$ in $X$ such that for each $i$, $\mathrm{tp}_X(a_i|\langle A \cup \{a_j : j < i\}\rangle)$ is $I$-wide. A theorem of Morley then yields an $A$-indiscernible sequence $(c_i)_{i<\omega+2}$ in $X$ such that for any $n$, $\mathrm{tp}((c_0, \dots, c_n)|A) = \mathrm{tp}((a_{i_0}, \dots, a_{i_n})|A)$ for suitable $i_0 < \dots < i_n < \beth_{\omega_1}$. In particular, $\mathrm{tp}_X(c_n|Ac_0 \dots c_{n-1})$ is $I$-wide for all $n$.

Let $F$ be a non-principal ultrafilter on $\mathbb{N} = \omega$ and put

$$\mathbf{p} := \{P \in \mathrm{Def}(X) : \{n : c_n \in P\} \in F\}.$$

Then $\mathbf{p} \in \mathrm{St}(X)$, and $\mathbf{p}$ is finitely satisfiable in $M := \langle A \cup \{c_i : i < \omega\}\rangle$. Put $a := c_{\omega+1}$ and $b := c_\omega$. Then $a \models \mathbf{p} \restriction M$, $b \models \mathbf{p} \restriction Ma$, and $\mathrm{tp}_X(a|Mb)$ is $I$-wide. This finishes the proof under the assumptions introduced earlier.

To reduce the general case to this special case, expand $\mathbb{U}$ to $\mathbb{U}_{\rm sk}$ while keeping the language countable and without introducing new sorts, such that $\mathrm{Th}(\mathbb{U}_{\rm sk})$ has definable Skolem functions. We cannot expect $\mathbb{U}_{\rm sk}$ to be big, so we take a big elementary extension $\mathbb{U}'$ of $\mathbb{U}_{\rm sk}$ with $\kappa(\mathbb{U}') > \beth_{\omega_1}$. If an $L(U)$-formula $\phi(x)$ defines $P \subseteq \mathbb{U}_x$ in $\mathbb{U}$, we denote the subset of $\mathbb{U}'_x$ that it defines in $\mathbb{U}'$ by $P'$. The $A$-open set $X \subseteq \mathbb{U}_x$ is likewise extended to an $A$-open set $X' \subseteq \mathbb{U}'_x$ (with respect to the ambient $\mathbb{U}'$) as follows:

$$\text{if } \ X = \bigcup_{j\in J} X_j, \quad \text{all } X_j \in \mathrm{Def}(\mathbb{U}_x|A), \ \text{ then } X' := \bigcup_{j\in J} X'_j.$$

Finally, let $I'$ be the collection of all sets $Q \subseteq (\mathbb{U}'_x$ that are definable in $\mathbb{U}'$ and contained in $P'$ for some $P \in I$. Then $I'$ is a proper $A$-invariant ideal of $\mathrm{Def}(X')$ (with respect to the ambient $\mathbb{U}'$). It remains to apply the result of the special case with $\mathbb{U}'$, $X'$, $I'$ in the role of $\mathbb{U}$, $X$, $I$, and restrict suitably. □

## 2. The stabilizer

In the first two subsections, *Generic sets*, and *The main theorem*, we work in $\mathbb{U}$ as before, and assume that $\kappa(\mathbb{U}) > 2^{\aleph_0}$ and $x$ is a *finite*[3] multivariable. We fix an $A$-definable group $G \subseteq \mathbb{U}_x$, that is, $G$ is an $A$-definable subset of $\mathbb{U}_x$ equipped with an $A$-definable group operation. We also assume given an $A$-definable subset $X$ of $G$ with $1_G \in X$. Put $X_1 := X^{-1}X$, so $X \subseteq X_1$ and $X_1^{-1} = X_1$, and let $X_n$ be the set of products $g_1 \cdots g_n$ with $g_1, \dots, g_n \in X_1$, so $X_n \subseteq X_{n+1}$, and $\widehat{X} := \bigcup_n X_n$ is the subgroup of $G$ generated by $X$. Each $X_n$ is $A$-definable, so $\widehat{X}$ is $A$-open. Finally, we fix an ideal $I$ of $\mathrm{Def}(\widehat{X})$ such that $X \notin I$, and $I$ is right-invariant: $Pg \in I$ for all $P \in I$ and $g \in X_1$ (so $Pg \in I$ for all $P \in I$ and $g \in \widehat{X}$).

In the last subsection *Applications* we derive some combinatorial results on finite subsets of groups in the spirit of the sum-product phenomena. The point here is to construct from an infinite sequence of counterexamples a situation where the main theorem can be applied.

**Generic sets.** The next lemma holds for any two subsets of any group.

**Lemma 2.1.** *Let $Y, Z \subseteq G$ and let $E$ be a maximal subset of $Z$ such that $Ye \cap Yf = \emptyset$ for all distinct $e, f \in E$. Then*

$$Z \subseteq \bigcup_{e\in E} Y^{-1}Ye.$$

*Proof.* Given $g \in Z$, the maximality of $E$ gives $e \in E$ such that $Yg \cap Ye \neq \emptyset$, so $ag = be$ with $a, b \in Y$, and thus $g \in Y^{-1}Ye$. □

[3]This is because we use a fact from [2] established there under this restriction.

A *right-generic set* is a set $Y \in \mathrm{Def}(\widehat{X})$ such that every $Z \in \mathrm{Def}(\widehat{X})$ is covered by finitely many right translates $Yg$ with $g \in \widehat{X}$, equivalently, every set $X_n$ is covered by finitely many such translates. If $Y$ is right-generic, then clearly $Y \notin I$. As a partial converse we have the following:

**Corollary 2.2.** *Suppose $I$ is S1 over $A$ and let $Y \in \mathrm{Def}(\widehat{X})$, $Y \notin I$. Then $Y^{-1}Y$ is right-generic.*

*Proof.* It suffices to show that each $X_n$ is covered by finitely many translates $Y^{-1}Ye$ with $e \in X_n$. Let $E$ be a maximal subset of $X_n$ such that $Ye \cap Yf = \emptyset$ for all distinct $e, f \in E$. Since $Ye \notin I$ for all $e \in E$, the assumption that $I$ is S1 over $A$ yields that $E$ is finite. Now apply Lemma 2.1. □

Recall from [2] that a countably definable subgroup of $G$ is a subgroup $H$ of $G$ whose underlying set is a countable intersection of definable subsets of $\mathbb{U}_x$. Suppose $H$ is a countably definable subgroup of $G$. Then by Lemma 4.5 of [2] we have $H = \bigcap_n H_n$ for a decreasing sequence $(H_n)$ of definable subsets of $G$ such that for all $n$,

$$H_n^{-1} = H_n, \qquad H_n \supseteq H_{n+1}H_{n+1}.$$

Note that if $H \subseteq \widehat{X}$, then $H \subseteq X_m$ for some $m$ (by compactness) and for such $m$ we have $H_n \subseteq X_m$ for all sufficiently large $n$, again by compactness.

**Lemma 2.3.** *Let $H$ be a countably definable subgroup of $\widehat{X}$. Then:*

(1) *$|\widehat{X}/H| \leq 2^{\aleph_0}$ iff every $Y \in \mathrm{Def}(\widehat{X})$ containing $H$ is right-generic;*
(2) *If $I$ is S1 over $A$, then $|\widehat{X}/H| \leq 2^{\aleph_0}$ iff $H$ is $I$-wide.*

*Proof.* Take a sequence $(H_n)$ as above with $H_n \subseteq \widehat{X}$ for all $n$. Let $m$ and $n > 0$ be such that $H_n \subseteq X_m$, and let $E$ be a maximal subset of $X_m$ such that $H_n e \cap H_n f = \emptyset$ for all distinct $e, f \in E$. By Lemma 2.1 we have $X_m \subseteq \bigcup_{e \in E} H_{n-1}e$. If $E$ is infinite, then by saturation $|E| > 2^{\aleph_0}$.

After these preliminary remarks, we prove the various parts of the lemma. If $|\widehat{X}/H| \leq 2^{\aleph_0}$, then by these remarks any $E$ as above is finite, and so by increasing $m$ and $n$ we see that all $H_n$ are right-generic, and thus every $Y \in \mathrm{Def}(\widehat{X})$ containing $H$ is right-generic.

Next, suppose that $|\widehat{X}/H| > 2^{\aleph_0}$. Take $F \subseteq \widehat{X}$ with $|F| > 2^{\aleph_0}$ such that $He \cap Hf = \emptyset$ for all distinct $e, f \in F$. We can arrange that for a certain $m$ we have $F \subseteq X_m$. For any distinct $e, f \in F$ we have $n = n(e, f) \in \mathbb{N}$ such that $H_n e \cap H_n f = \emptyset$. By Erdös-Rado, we obtain an infinite subset $E$ of $F$ such that $n(e, f)$ takes a constant value $n$ for distinct $e, f \in E$. Increasing $m$ if necessary we can assume that $H_n \subseteq X_m$. We claim that then $H_n$ is not right-generic. Otherwise we have $E \subseteq X_m \subseteq H_n e_1 \cup \cdots \cup H_n e_k$ with $k \in \mathbb{N}$ and $e_1, \ldots, e_k \in \widehat{X}$, so we get $i \in \{1, \ldots, k\}$ and distinct $e, f \in E$ such that $e = ge_i$ and $f = he_i$ with $g, h \in H_n$, and thus $g^{-1}e = h^{-1}f \in H_n e \cap H_n f$, a contradiction.

We now prove the second statement, so assume $I$ is S1 over $A$. If $H$ is contained in some $P \in I$, then some $H_n$ is contained in some $P \in I$, and thus not right-generic, so $|\widehat{X}/H| > 2^{\aleph_0}$ by the first part. Conversely, assume $|\widehat{X}/H| > 2^{\aleph_0}$. Then the first part gives $n$ such that $H_n$ is not right-generic, so $H_{n+1}^{-1}H_{n+1}$ is not right-generic, and thus $H_{n+1} \in I$ by Corollary 2.2. □

With an eye on applying this lemma we note that if $L$ and $A$ are countable, then every $A$-closed subgroup of $G$ is countably definable.

**The main result.** Recall our standing assumption that $I$ is right-invariant. The next theorem is a major result and assumes also left-invariance:

**Theorem 2.4.** *Assume $A \subseteq M$, $I$ is S1 over $M$ and left-invariant: $gP \in I$ for all $P \in I$ and $g \in X_1$. Let $q \in \mathrm{St}(X|M)$ be $I$-wide and assume there are $a, b \in [q] := q(X)$ such that $\mathrm{tp}(b|Ma)$ and $\mathrm{tp}(a|Mb)$ do not fork over $M$.*

*Then $H := [q]^{-1}[q][q]^{-1}[q]$ is an $I$-wide $M$-closed normal subgroup of $\widehat{X}$, the set $H \setminus [q]^{-1}[q]$ is contained in a union of $M$-definable sets in $I$, and $[q][q]^{-1}[q]$ is a coset of $H$ in $\widehat{X}$.*

*Proof.* For simplicity we assume that $L$ and $M$ are countable. (This doesn't really affect applications, and when time permits I will add a reduction of the general case to this special case.) For sets $Y \subseteq \mathbb{U}_y$ and $Z \subseteq \mathbb{U}_z$ we put

$$Y \times_{\mathrm{nf}} Z := \{(a, b) \in Y \times Z : \ \mathrm{tp}(b|Ma) \text{ does not fork over } M\}.$$

Next we introduce $M$-invariant subsets $Q$ and $Q'$ of $X_1$:

$$Q := \{a^{-1}b : \ (a, b) \in [q] \times_{\mathrm{nf}} [q]\},$$
$$Q' := \{a^{-1}b : \ a, b \in [q], \ \mathrm{tp}_X(b|Ma) \text{ is } I\text{-wide}\}.$$

We have $Q \supseteq Q'$ by Lemma 1.18, and $[q][q]^{-1}$ is $I$-wide by right-invariance of $I$, and $[q]^{-1}[q]$ is $I$-wide by the left-invariance of $I$. Note also that we have a type $q^{-1} \in \mathrm{St}(X^{-1}|M)$ such that $[q]^{-1} = q^{-1}(X^{-1})$. Throughout the proof we use that by Lemma 1.19 the relation $R \subseteq X_2 \times X_2$ given by

$$R(a, b) \iff [q]a^{-1} \cap [q]b^{-1} \text{ is } I\text{-wide},$$

is stable over $M$. Moreover, by Lemma 1.14 and the symmetry of $R$, given any types $p, p' \in \mathrm{St}(X_2|M)$, the following are equivalent:

(i) $R(a, b)$ for some $(a, b) \in p(X_2) \times_{\mathrm{nf}} p'(X_2)$;
(ii) $R(a, b)$ for all such $(a, b)$;
(iii) $R(a, b)$ for some $(a, b) \in p'(X_2) \times_{\mathrm{nf}} p(X_2)$;
(iv) $R(a, b)$ for all such $(a, b)$.

It follows in particular that if $Y, Z \subseteq X_2$ are $M$-invariant and $R(a, b)$ for all $(a, b) \in Y \times_{\mathrm{nf}} Z$, then also $R(a, b)$ for all $(a, b) \in Z \times_{\mathrm{nf}} Y$.

**Claim 1.** $[q]^{-1}[q] \subseteq QQ$.

To prove this, let $a, b \in [q]$. By the assumption on $q$ there is $c \in [q]$ such that $\mathrm{tp}(a|Mc)$ and $\mathrm{tp}(c|Ma)$ do not fork over $M$. By Lemma 1.6 we can extend $\mathrm{tp}(c|Ma)$ to some $p \in \mathrm{St}(\mathbb{U}_x|Mab)$ that doesn't fork over $M$, so that upon replacing $c$ by an element of $p(\mathbb{U}_x)$ we arrange that $\mathrm{tp}(c|Mab)$ doesn't fork over $M$. Then $(b,c),(c,a) \in [q] \times_{\mathrm{nf}} [q]$, so $b^{-1}c, c^{-1}a \in Q$, and thus $b^{-1}a \in QQ$.

**Claim 2.** $R(a,b)$ for all $(a,b) \in [q] \times_{\mathrm{nf}} [q]$.

By the remarks on $R$ preceding Claim 1 it is enough to show that $R(a,b)$ for some $(a,b) \in [q] \times_{\mathrm{nf}} [q]$. By the remark following the proof of Lemma 0.2 we can extend $q$ to an $M$-invariant global type $\mathbf{q} \in \mathrm{St}(X)$. Take a $\mathbf{q}$-indiscernible sequence $(a_n)$ over $M$. Then $(a_m, a_n) \in [q] \times_{\mathrm{nf}} [q]$ for all $m < n$. For $P \in q$ we have $P \notin I$, so $Pa_n^{-1} \notin I$ for all $n$; since $I$ is S1 over $M$ we get $m < n$ such that $Pa_m^{-1} \cap Pa_n^{-1} \notin I$, and thus $Pa_0^{-1} \cap Pa_1^{-1} \notin I$. This holds for all $P \in q$, so $[q]a_0^{-1} \cap [q]a_1^{-1}$ is $I$-wide, that is, $R(a_0, a_1)$.

**Claim 3.** $R(c,c')$ for all $(c,c') \in [q]^{-1}[q] \times_{\mathrm{nf}} Q'$.

Let $(c,c') \in [q]^{-1}[q] \times_{\mathrm{nf}} Q'$, and $p := \mathrm{tp}(c|M)$ and $p' := \mathrm{tp}(c'|M)$. As before it is enough to show that $R(d,d')$ for some $(d,d') \in p(X_1) \times_{\mathrm{nf}} p'(X_1)$. Let $a_0 \in [q]$, and take $a_1 \in [q]$ such that $\mathrm{tp}(a_0^{-1}a_1|M) = \mathrm{tp}(c|M) = p$. From $c' \in Q'$ we get $a_2' \in [q]$ such that $\mathrm{tp}(a_0^{-1}a_2'|M) = \mathrm{tp}(c'|M) = p'$ and $r := \mathrm{tp}_X(a_2'|Ma_0)$ is $I$-wide. Extend $r$ to an $I$-wide $r' \in \mathrm{St}(X|Ma_0a_1)$ and take $a_2 \in r'(X) \subseteq q(X) = [q]$. Then

$$\mathrm{tp}(a_0^{-1}a_1|M) = p, \qquad \mathrm{tp}(a_0^{-1}a_2|M) = p',$$

and $\mathrm{tp}_X(a_2|Ma_0a_1) = r'$ is $I$-wide, hence doesn't fork over $M$ by Lemma 1.18, and so $\mathrm{tp}(a_2|Ma_1)$ doesn't fork over $M$. Then $[q]a_1^{-1} \cap [q]a_2^{-1}$ is $I$-wide by Claim 2, so $[q]a_1^{-1}a_0 \cap [q]a_2^{-1}a_0$ is $I$-wide by the right-invariance of $I$, and thus $R(a_0^{-1}a_1, a_0^{-1}a_2)$.

**Claim 4.** $R(c,d)$ for all $(c,d) \in [q]^{-1}[q] \times_{\mathrm{nf}} Q$.

To prove this, let $(c,d) \in [q]^{-1}[q] \times_{\mathrm{nf}} Q$. Then $d = a^{-1}b$ where $a,b \in [q]$ and $\mathrm{tp}(b|Ma)$ doesn't fork over $M$. We wish to show $R(c, a^{-1}b)$, which by the right-invariance of $I$ is equivalent to $R(ac,b)$. Since $\mathrm{tp}_X(b|M) = q$ is $I$-wide, Lemma 1.20 provides $b' \in [q]$ such that $\mathrm{tp}_X(b'|M) = \mathrm{tp}_X(b|M)$ and $\mathrm{tp}_X(b'|Mac)$ is $I$-wide, and so $\mathrm{tp}(b'|M(ac))$ doesn't fork over $M$ by Lemma 1.18. By the remarks preceding Claim 1 it suffices to show that $R(ac,b')$, equivalently, $R(c,a^{-1}b')$. Now $\mathrm{tp}_X(b'|Ma)$ is $I$-wide, so $a^{-1}b' \in Q'$. Left-invariance of $I$ gives that $\mathrm{tp}_{X_1}(a^{-1}b'|Mac)$ is $I$-wide, so $\mathrm{tp}_{X_1}(a^{-1}b'|Mc)$ is $I$-wide, and thus $\mathrm{tp}(a^{-1}b'|Mc)$ doesn't fork over $M$. Now Claim 3 yields $R(c,a^{-1}b')$, as desired.

**Claim 5.** Let $(b,a) \in Q \times_{\mathrm{nf}} [q]^{-1}[q]$. Then $[q]a \cap [q]b^{-1}$ is $I$-wide and (thus) $ab \in [q]^{-1}[q]$.

To see this, note that $(b,a^{-1}) \in Q \times_{\mathrm{nf}} [q]^{-1}[q]$, and thus $R(b,a^{-1})$ by Claim 4 and the symmetry property of $R$ mentioned just before Claim 1. Hence $[q]b^{-1} \cap [q]a$ is $I$-wide, and so we can take $c,d \in [q]$ with $cb^{-1} = da$, and thus $ab = d^{-1}c \in [q]^{-1}[q]$.

**Claim 6.** Let $a \in [q]^{-1}[q]$, $n \geq 1$, and $b_1, \ldots, b_n \in Q$ be such that the type $\mathrm{tp}_{X_1}(a|Mb_1 \ldots b_n)$ is $I$-wide. Then $ab_1 \cdots b_n \in [q]^{-1}[q]$ and the set $[q]a \cap [q](b_1 \cdots b_n)^{-1}$ is $I$-wide.

We prove this by induction on $n$. Since $\mathrm{tp}_{X_1}(a|Mb_1)$ is $I$-wide, it doesn't fork over $M$ by Lemma 1.18, and so by Claim 5,

$$ab_1 \in [q]^{-1}[q], \qquad [q]a \cap [q]b_1^{-1} \text{ is } I\text{-wide.}$$

This gives the case $n = 1$. Let $n > 1$. Since $\mathrm{tp}_{X_1}(a|Mb_1 \ldots b_n)$ is $I$-wide, the right-invariance of $I$ yields that $\mathrm{tp}_{X_1}(ab_1|Mb_1 \ldots b_n)$ is $I$-wide, and therefore $\mathrm{tp}_{X_1}(ab_1|Mb_2 \ldots b_n)$ is $I$-wide. Then by the inductive assumption,

$$ab_1(b_2 \cdots b_n) \in [q]^{-1}[q] \text{ and } [q]ab_1 \cap [q](b_2 \cdots b_n)^{-1} \text{ is } I\text{-wide.}$$

But $I$ is right-invariant, so $[q]a \cap [q](b_1 \cdots b_n)^{-1}$ is $I$-wide.

**Claim 7.** Let $n \geq 1$. Then $\{b_1 \cdots b_n : b_1, \ldots, b_n \in Q\} \subseteq [q]^{-1}[q][q]^{-1}[q]$.

Let $b_1, \ldots, b_n \in Q$. Now $[q]^{-1}[q] \subseteq X_1$ is $I$-wide, so $[q]^{-1}[q] \in p$ for some $I$-wide $p \in \mathrm{St}(X_1|Mb_1 \ldots b_n)$, and then $a \in p(X_1)$ has the property that $a \in [q]^{-1}[q]$ and $\mathrm{tp}(a|Mb_1 \ldots b_n)$ is $I$-wide. Then $ab_1 \cdots b_n \in [q]^{-1}[q]$ by Claim 6, so

$$b_1 \cdots b_n = a^{-1}(ab_1 \cdots b_n) \in [q]^{-1}[q][q]^{-1}[q].$$

Claims 1 and 7 yield that $H := [q]^{-1}[q][q]^{-1}[q]$ is indeed a subgroup of $\widehat{X}$. Clearly $H$ is $M$-closed, $H \subseteq X_2 \subseteq \widehat{X}$, and $H \supseteq [q]^{-1}[q]$, so $H$ is $I$-wide.

**Claim 8.** Let $T$ be an $M$-closed subgroup of $H$ such that $|H/T| < \kappa(\mathbb{U})$. Then $T = H$.

To prove this, first note that $[q]^{-1}[q] \subseteq H$ gives that $[q]$ is contained in a single left coset of $H$ in $\widehat{X}$, and this left coset equals $[q]H$, and is therefore $M$-closed. We have $T$ acting on $[q]H$ by multiplication on the right, and this action has only a small number of orbits $aT$ with $a \in [q]H$. Then by Lemma 1.17 we have $[q] \subseteq aT$ where $a \in [q]$, and thus $[q]^{-1}[q] \subseteq T$, which gives $H = T$.

**Claim 9.** The group $H$ is normal in $\widehat{X}$.

Let $a \in \widehat{X}$, say $a \in X_n$, and put $r := \mathrm{tp}(a|M)$, so $r(\mathbb{U}_x) \subseteq X_n$. Hence $r(\mathbb{U}_x)$ is contained in the union of the small number of left cosets of $H$ in $G$ that meet $X_n$. Then it follows from Lemma 1.17 that $r(\mathbb{U}_x)$ is contained in a single such coset, which must equal $aH$ and must be $M$-closed. Hence $H' := aH \cdot H \cdot (aH)^{-1} = aHa^{-1}$ is $M$-closed. By Lemma 2.3 and the remark following its proof we have

$$|\widehat{X}/H'| = |\widehat{X}/H| \leq 2^{\aleph_0}.$$

(This is the only place in the proof of the theorem where we use that $L$ and $M$ are countable.) It follows that $|H/H \cap H'| < \kappa(\mathbb{U})$, and thus $H \subseteq H'$ by

Claim 8, that is, $H \subseteq aHa^{-1}$, so $a^{-1}Ha \subseteq H$. This holds for all $a \in \widehat{X}$, so $H$ is normal in $\widehat{X}$.

**Claim 10.** The sets $Q$ and $Q'$ are $I$-wide.

To prove this claim, take $a \in [q]$ and use Lemma 1.20 to extend $q$ to an $I$-wide type $p \in \mathrm{St}(X|Ma)$. If $b \in p(X)$, then $\mathrm{tp}_X(b|Ma) = p$ is $I$-wide, so $a^{-1}b \in Q'$. Hence $a^{-1}p(X) \subseteq Q'$. By the left-invariance of $I$ the set $a^{-1}p(X)$ is $I$-wide, and so $Q'$ is $I$-wide. Thus $Q \supseteq Q'$ is $I$-wide.

**Claim 11.** $H \setminus [q]^{-1}[q]$ is contained in a union of $M$-definable sets in $I$.

To prove this, note first that the set $H \setminus [q]^{-1}[q] \subseteq X_2$ is $M$-invariant, and thus a union of sets $r(X_2)$ with $r \in \mathrm{St}(X_2|M)$. So let any $I$-wide $r \in \mathrm{St}(X_2|M)$ with $r(X_2) \subseteq H$ given; it suffices to show that then

$$r(X_2) \subseteq [q]^{-1}[q].$$

Pick $s_0 \in r(X_2)$, and $c_0 \in Q$. By Claim 10, $\mathrm{tp}_{X_1}(c_0|M)$ is $I$-wide, and so by Lemma 1.20 we get $c \in Q$ such that $\mathrm{tp}_{X_1}(c|M) = \mathrm{tp}_{X_1}(c_0|M)$ and $\mathrm{tp}_{X_1}(c|Ms_0)$ is $I$-wide. Since $c \in [q]^{-1}[q]$ and $s_0^{-1} \in H \subseteq QQQQ$ it follows from Claim 6 that $[q]c \cap [q]s_0$ is $I$-wide.

Since $r$ is $I$-wide we get $s \in r(X_2)$ such that $\mathrm{tp}_{X_2}(s|Mc)$ is $I$-wide, so $\mathrm{tp}(s|Mc)$ doesn't fork over $M$, and thus $[q]c \cap [q]s$ is $I$-wide as well, using the remarks on $R$ preceding Claim 1. Hence $sc^{-1} \in [q]^{-1}[q]$. The right-invariance of $I$ gives that $\mathrm{tp}_{X_1}(sc^{-1}|Mc)$ is $I$-wide, and so $\mathrm{tp}(sc^{-1}|Mc)$ doesn't fork over $M$. Hence $(c, sc^{-1}) \in Q \times_{\mathrm{nf}} [q]^{-1}[q]$, and so by Claim 5 we have $s = (sc^{-1})c \in [q]^{-1}[q]$, and thus $r(X_2) \subseteq [q]^{-1}[q]$.

**Claim 12.** $[q][q]^{-1}[q] = aH$ for all $a \in [q]$.

Let $a \in [q]$ and note that in the proof of Claim 8 we obtained $[q] \subseteq aH$, and so $[q][q]^{-1}[q] \subseteq aH$. Next, let $b \in aH$. Then $b = ab_1b_2b_3b_4$ with $b_1, b_2, b_3, b_4 \in Q$ by Claim 1. Take $c \in [q]^{-1}[q]$ such that $\mathrm{tp}(c|Mb_1b_2b_3b_4)$ is $I$-wide. Then $cb_1b_2b_3b_4 \in [q]^{-1}[q]$ by Claim 6, so

$$b = ab_1b_2b_3b_4 = ac^{-1}cb_1b_2b_3b_4 \;\in\; [q][q]^{-1}[q][q]^{-1}[q] = [q]H = aH.$$

This concludes the proof of Claim 12 and of the theorem. □

We call attention to Claim 8 in the proof above, which says that there is no $M$-closed proper subgroup $T$ of $H$ with small quotient $H/T$. Of course, it also follows from the $I$-wideness of $H$ and Lemma 2.3 that $|\widehat{X}/H| \leq 2^{\aleph_0}$ if $L$ and $M$ are countable.

It is also worth pointing out that $H \subseteq X_1 \cup Y$ for some $M$-definable $Y \in I$: "almost all" elements of $H$ are in $X_1$. This is because $H$ is contained in $[q]^{-1}[q] \cup \bigcup_{\lambda\in\Lambda} Y_\lambda \subseteq X_1 \cup \bigcup_{\lambda\in\Lambda} Y_\lambda$ where all $Y_\lambda$ are $M$-definable and in $I$. Now use compactness of the $M$-topology to get a single $Y$.

**On the use of left-invariance.** The first use of the assumption that $I$ is left-invariant in the proof above is in asserting that $[q]^{-1}[q]$ is $I$-wide. But this assertion holds without assuming left-invariance: let $Y \in q$ and take a maximal subset $E$ of $[q]$ such that $Ye \cap Yf = \emptyset$ for all distinct $e, f \in E$. Then $E$ is finite since $Y \notin I$ and $I$ is S1 over $M$. Also $[q] \subseteq \bigcup_{e \in E} Y^{-1}Ye$ by Lemma 2.2. Hence $Y^{-1}Y \notin I$, for every $Y \in q$, so $[q]^{-1}[q]$ is $I$-wide.

Other uses of left-invariance in proving Theorem 2.4 are in establishing Claims 4 and 10. (Hrushovski's argument for Claim 10 is not clear to me.)

**Remarks.** Lemmas 1.22, 1.23, 1.24 are useful in getting the hypotheses of Theorem 2.4 satisfied. More precisely:

(1) Suppose $A \subseteq M$, and $I$ is $M$-open, S1 over $M$, and left-invariant. Then there is an $I$-wide $q \in \mathrm{St}(X|M)$ with $a, b \in q(X)$ such that $\mathrm{tp}(a|Mb)$ and $\mathrm{tp}(b|Ma)$ don't fork over $M$. (Note that for such $q$ the assumptions of Theorem 2.4 are satisfied.)

(2) Suppose $L$ and $A$ are countable, and $I$ is S1 over $A$, and left-invariant. Then there exists a countable $M \supseteq A$ and an $I$-wide $q \in \mathrm{St}(X|M)$ with $a, b \in q(X)$ such that $\mathrm{tp}(a|Mb)$ and $\mathrm{tp}(b|Ma)$ don't fork over $M$. (Note that for such $M$ and $q$ the assumptions of Theorem 2.4 are satisfied.)

Recall that $I$ is proper by our standing assumption that $X \notin I$. Thus (1) follows from Lemma 1.22 and (2) from Lemma 1.24.

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