Let $f$ be a transformation of the plane. We say that a point $P$ is a fixed point of $f$ is $f(P) = P$. Let’s explore the properties of $f$ as they relate to its fixed points.

Assume $f$ is an isometry throughout.

1. Show that if $A, B$ are two different fixed points of $f$, then every point on the line $AB$ is also a fixed point of $f$.

An isometry of this sort, i.e. one which has two different fixed points $A, B$, is called a reflection. The line $AB$ is called the axis of reflection of $f$.

2. Suppose $f$ is a reflection about an axis $AB$, and $P$ is a point not on the line $AB$. Assume $P$ is not fixed by $f$. What can you say about the relationship between the line $AB$ and the line $Pf(P)$? First guess some claims, then prove them.

Be careful: do not assume, unless you prove it first, that $f$ “flips” the halves of the plane determined by $AB$.

3. Given an axis $AB$, how many different reflections about $AB$ are there? Given a point $P$, how can you construct the image of $P$ under a reflection?

4. Next suppose we are given two different points $P, Q$, but no $f$, and ask whether we can always find/define a reflection $f$ for which $Q = f(P)$? What do you think? Guess an answer and then prove it.

5. Now go back again to $f$ being any isometry, not necessarily a reflection. Show that if $A, B, C$ are three non-collinear points, all fixed by $f$, then $f$ is the identity.

6. Show that if $f$ and $g$ are two isometries that agree on three non-collinear points, then they agree on all points.