The cross-ratio of four complex numbers $z_0, z_1, z_2, z_3$ is defined as

$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_3}{z_0 - z_1} \frac{z_2 - z_3}{z_2 - z_1}.$$

(a). Suppose $z_1, z_2, z_3$ are three distinct complex numbers, and let $f$ be a Möbius transformation. Prove that for any $z$,

$$(f(z), f(z_1), f(z_2), f(z_3)) = (z, z_1, z_2, z_3).$$

**Hint:** For the function $g(z) = (z, z_1, z_2, z_3)$, analyze the effect of $g \circ f^{-1}$ on $f(z_1), f(z_2), f(z_3)$. How does that compare to the function $h(z) = (z, f(z_1), f(z_2), f(z_3))$?

(b). In this part, you will follow the outline steps to prove the following theorem: Let $z_0, z_1, z_2, z_3$ be four distinct points. Then the cross ratio $(z_0, z_1, z_2, z_3)$ is a real number if and only if the four points lie on a cline.

- To start the proof, you use that three distinct points define a unique cline. So, $z_1, z_2, z_3$ define a unique cline, and the question is how to tell if $z_0$ is on it or not. What could the equation for a cline through $z_1, z_2, z_3$ look like?
- Show that if $z_0$ satisfies such an equation, then the cross ratio is real.
- Now you move on to the converse. Study the function

$$f(z) = (z, z_1, z_2, z_3).$$

How can you tell if $f(z)$ is real?