

MA225 (T2) SOLUTIONS TO HOMEWORK 1

Section 1.1

Problem 13: The given system of equations can be written as an augmented matrix as follows:

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix}$$

If we row reduce by the following row operations:

$$\begin{aligned} R_2 &\rightarrow R_2 - 2R_1 - R_3 \\ R_3 &\rightarrow R_3 - R_2 \end{aligned}$$

we get the matrix

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 10 & -7 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

which gives

$$\begin{aligned} x_1 - 3x_3 &= 8 \\ x_2 + 10x_3 &= -7 \\ -5x_3 &= 5 \end{aligned}$$

The third equation gives $x_3 = -1$, substituting this into the first and second equations gives $x_2 = 3$ and $x_1 = 5$

Problem 21: By the single row operation $R_2 \rightarrow R_2 + 4R_1$ the given system can be reduced to:

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & h + 12 & 0 \end{bmatrix}$$

The first row can be solved to get x_1 in terms of x_2 so for consistency we focus on the second row which gives the equation $(h + 12)x_2 = 0$. This has a valid solution $x_2 = 0$ for any value of h .

Section 1.2

Problem 12: By the row operation $R_3 \rightarrow R_3 + R_1 + 4R_2$ the given augmented matrix reduces to:

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Choose x_4 and x_2 to be free variables i.e $x_4 = t$ and $x_2 = s$ then we get

$$\begin{aligned} x_3 &= 2t - 3 \\ x_1 &= 7s - 6t + 5 \end{aligned}$$

Problem 16(b): The system is consistent since all rows have a nonzero pivot (in general, for consistency either a row should have a pivot or in the absence of a pivot all its entries should be zero). The solution is not unique because x_2 can be chosen to be a free parameter (in fact, x_3 and x_4 are completely determined, but x_2 is a free parameter and x_1 can be expressed in terms of this free parameter)

Section 1.3

Problem 17: \mathbf{b} is in the plane spanned by \mathbf{a}_1 and \mathbf{a}_2 if and only if \mathbf{b} is a linear combination of \mathbf{a}_1 and \mathbf{a}_2 i.e there exists real numbers x_1 and x_2 such that $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$. If we restrict to the first two rows of this vector equation we get:

$$x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

which can be rewritten as

$$\begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Solving this by row reduction on the augmented matrix we get $x_1 = -2$ and $x_2 = -3$. Plugging these values back into the vector equation $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$ and restricting to the third row we get $h = -17$

Problem 26:

- (1) $\mathbf{b} \in W$ if and only if \mathbf{b} is a linear combination of the columns

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix} \quad \mathbf{a}_3 = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$$

i.e there exists real numbers x_1, x_2 and x_3 such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$
A clever way is to notice that $x_1 = 2, x_2 = 0$ and $x_3 = 1$ works. The systematic way to do this is to rewrite the vector equation as $\mathbf{Ax} = \mathbf{b}$ i.e

$$\begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$$

and the question whether $\mathbf{b} \in W$ becomes equivalent to whether the above system is consistent. Applying the row reductions $R_3 \rightarrow 8R_3 + 2R_2 - 3R_1, R_2 \rightarrow 1/16(2R_2 + R_1)$ on the augmented matrix we get:

$$\begin{bmatrix} 2 & 0 & 6 & 10 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This shows that the system is consistent and choosing x_3 as a free parameter x_1 and x_2 can be written in terms of x_3 . Thus we conclude that $\mathbf{b} \in W$

- (2) The third column of \mathbf{A} i.e \mathbf{a}_3 is in W because it is a linear combination in the obvious way viz $\mathbf{a}_3 = 0\mathbf{a}_1 + 0\mathbf{a}_2 + 1\mathbf{a}_3$

Section 1.4

Problem 16: Row reduction on the augmented matrix by $R_3 \rightarrow R_3 + R_1 + 2R_2, R_2 \rightarrow R_2 + 3R_1$ gives

$$\begin{bmatrix} 1 & -3 & -4 & & b_1 \\ 0 & -7 & -6 & & b_2 + 3b_1 \\ 0 & 0 & 0 & b_3 + b_1 + 2b_2 & \end{bmatrix}$$

Restricting to the third row of the above matrix we see that for the solution to exist (i.e consistency) we must have the entry $b_3 + 3b_1 + 2b_2 = 0$, this determines a plane through the origin (b_2 and b_3 can be chosen to be free to parameterize the plane) There is no solution when $b_3 + 3b_1 + 2b_2$ is non-zero.

Problem 26: The equation $3\mathbf{u} - 5\mathbf{v} - \mathbf{w} = 0$ is equivalent to $3\mathbf{u} - 5\mathbf{v} = \mathbf{w}$. This can now be rewritten as

$$\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} = [\mathbf{w}]$$

that is

$$\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$$

Thus $x_1 = 3$ and $x_2 = -5$

Problem 33: The first point to note is that if for some $\mathbf{b} \in \mathbb{R}^4$, the solution to $A\mathbf{x} = \mathbf{b}$ is unique, then for all $\mathbf{b} \in \mathbb{R}^4$ for which the solution exists, it is unique. Hence we can assume that A is in row reduced echelon form to begin with.

Lets suppose now that the number of pivots in A is less than 3. In this case, the pivots must occur in at most 2 of the columns leaving at least 1 column without a pivot. This implies that in the solution the variable x_i corresponding to the pivotless column i is a free variable. But once we have a free variable the solution can no longer be unique, since every value the free variable takes gives us a different solution. Thus we have proved that the number of pivots in A has to be 3. Once we know this, it is easy to write down the row reduced echelon form for A since there is only 1 choice for it, namely:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$