MA225 (T2) SOLUTIONS TO HOMEWORK 2

Section 1.5

Problem 15: Row reduction on the augmented matrix by the row operations $R_2 \longrightarrow 1/3(R_2 + 4R_1), R_3 \longrightarrow R_3 + 3R_2$ gives

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Thus x_3 can be chosen to be a free parameter t and solving for x_2 and x_1 in terms of t we get $x_2 = 1 - 2t$ and $x_1 = -2 + 5t$. Writing this down as a vector we have:

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 + 5t \\ 1 - 2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

The solution set is the line in $\mathbb{R}^3$ passing through the points $\mathbf{x}(0) = (-2, 1, 0)$ and $\mathbf{x}(1) = (3, -1, 1)$. Problem 5 solves the corresponding homogeneous equation. The solution set for the homogeneous equation is a line parallel to the one we get above and passing through the origin. It can be parameterized as simply

$$\mathbf{x}_h(t) = t \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

Its clear from the parametrization that $\mathbf{x}(t)$ is obtained from $\mathbf{x}_h(t)$ by a translation by the vector $(-2, 1, 0)$

Problem 18: In both cases, x_2 and x_3 can be chosen as free parameters s and t respectively. For $x_1 - 3x_2 + 5x_3 = 0$ the solution set is

$$\mathbf{x}_h = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3s - 5t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

Geometrically this is a plane through origin in $\mathbb{R}^3$ containing the points $(3, 1, 0)$ and $(-5, 0, 1)$. For $x_1 - 3x_2 + 5x_3 = 4$ the solution set is

$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \mathbf{x}_h$$

Geometrically this is a plane parallel to the one above obtained by translating the original one by the vector $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$

Problem 37: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $A \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4a + b \\ 4c + d \end{bmatrix}$ If we impose that the vector on the right is the zero vector then we must have $4a + b = 4c + d = 0$.

This can be achieved by choosing $a = 1$ and $b = -4$ and the pair $(c, d) = \lambda(a, b)$ for some real number λ. Thus our matrix $A = \begin{bmatrix} 1 & -4 \\ \lambda & -4\lambda \end{bmatrix}$ Now perform row reduction on the augmented matrix $\begin{bmatrix} 1 & -4 & b_1 \\ \lambda & -4\lambda & b_2 \end{bmatrix}$ by $R_2 \to R_2 - \lambda R_1$ to get $\begin{bmatrix} 1 & -4 & b_1 \\ 0 & 0 & b_2 - \lambda b_1 \end{bmatrix}$. If $b_2 - \lambda b_1 \neq 0$ then the equation $A\mathbf{x} = \mathbf{b}$ has no solution at all, so in particular it is not a translate of the solution of the homogeneous equation. This does not contradict Theorem 6 because the theorem requires that the solution set of $A\mathbf{x} = \mathbf{b}$ be non-empty to begin with.
Remark: A is a *linear transformation* from $\mathbb{R}^2$ to $\mathbb{R}^2$ i.e for each vector v in $\mathbb{R}^2$, it associates a vector Av in $\mathbb{R}^2$ such that linear combinations go to linear combinations i.e $\lambda_1 v_1 + \lambda_2 v_2$ gets sent to $\lambda_1(Av_1) + \lambda_2(Av_2)$. The A in the problem sends the vector $\mathbf{a} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ to the zero vector and all other vectors which do not lie on the line passing through the origin and $\mathbf{a}$ to some non-zero vector on the line passing through the origin and $\mathbf{b} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$. So roughly speaking, A "kills" or "collapses" everything in the $\mathbf{a}$ direction and maps everything in the "transverse" direction onto the $\mathbf{b}$ direction.

Section 1.7

Problem 13: Denote the 3 vectors by v_1, v_2 and v_3 respectively. They are linearly dependent if and only if the vector equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$ has a *non-trivial* solution. The vector equation can be equivalently written as $A\mathbf{x} = 0$ where

$$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 5 & -9 & h \\ -3 & 6 & -9 \end{bmatrix}$$

Row operation $R_3 \to R_3 + 3R_1, R_2 \to R_2 - 5R_1$ on the augmented matrix gives

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & h - 15 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this one can conclude that irrespective of the value of h, x_3 can be taken to be a free variable, thus the existence of non-trivial solution. Hence the vectors are linearly dependent for all values of h (The quickest way to notice this is to see that the third row of A is a multiple of the first row, thus A cannot be invertible and so there must exist a non-trivial solution)

Problem 39: Suppose that for all $\mathbf{b}$ in $\mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has at most one solution; so in particular $A\mathbf{x} = \mathbf{0}$ has at most one solution. The equation $A\mathbf{x} = \mathbf{0}$ always has the trivial solution. This means that it has no non-trivial solution, so columns of A have to be linearly independent.

Section 2.1

Problem 10: Compute

$$\begin{aligned} AB &= \begin{bmatrix} 2\cdot 8+(-3)\cdot 5 & 2\cdot 4+(-3)\cdot 5 \\ (-4)\cdot 8+6\cdot 5 & (-4)\cdot 4+6\cdot 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix} \\ AC &= \begin{bmatrix} 2\cdot 5+(-3)\cdot 3 & 2\cdot(-2)+(-3)\cdot 1 \\ (-4)\cdot 5+6\cdot 3 & (-4)\cdot(-2)+6\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix} \end{aligned}$$

which shows that $AB = AC$ but clearly $B \neq C$. The point of this problem is to show that matrix multiplication does not have the *left cancelation* property in general. $AB = AC$ implies $B = C$ only if A is invertible.

Problem 22: Suppose that the columns of B are linearly dependent. This is equivalent to the existence of a non-trivial solution for the equation $B\mathbf{x} = 0$. Lets call the solution $\mathbf{y}$. Then by the associativity of matrix multiplication $(AB)\mathbf{y} = A(B\mathbf{y}) = A\mathbf{0} = 0$ and so $\mathbf{y}$ is also a solution for the equation $(AB)\mathbf{x} = 0$. But this implies that the columns of AB are linearly dependent and we are done.