MA225 (T2) SOLUTIONS TO HOMEWORK 4

Section 3.3

Problem 5: Writing the system in matrix form we get:

$$\begin{bmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -8 \\ -3 \end{bmatrix}$$

The determinant of the coefficient matrix is 4 which is non-zero. So Cramer's rule applies and we get the solution set:

$$x_1 = \frac{1}{\det A} \det \begin{bmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{bmatrix} = \frac{6}{4} = \frac{3}{2}$$

$$x_2 = \frac{1}{\det A} \det \begin{bmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{bmatrix} = \frac{16}{4} = 4$$

$$x_3 = \frac{1}{\det A} \det \begin{bmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{bmatrix} = -\frac{14}{4} = -\frac{7}{2}$$

Problem 13: Compute the various co-factors:

$$\begin{aligned}
c_{11} &= (-1)^{1+1} \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = -1 \\
c_{12} &= (-1)^{1+2} \det \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = 1 \\
c_{13} &= (-1)^{1+3} \det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = 1 \\
c_{21} &= (-1)^{2+1} \det \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} = -1 \\
c_{22} &= (-1)^{2+2} \det \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} = -5 \\
c_{23} &= (-1)^{2+3} \det \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix} = 7 \\
c_{31} &= (-1)^{3+1} \det \begin{bmatrix} 5 & 4 \\ 0 & 1 \end{bmatrix} = 5 \\
c_{32} &= (-1)^{3+2} \det \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} = 1 \\
c_{33} &= (-1)^{3+3} \det \begin{bmatrix} 3 & 5 \\ 1 & 0 \end{bmatrix} = -5
\end{aligned}$$

The inverse is now given by:

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix}$$

Problem 21: Area of the parallelogram is invariant under translation. So translating all four vertices by the vector $(1,0)$ we get the parallelogram with the same area and vertices $(0,0)$, $(1,5)$, $(2,-4)$ and $(3,1)$. Adjacent sides of the parallelogram are given by the vectors $(2,-4)$ and $(1,5)$ in the counterclockwise order. The area of the parallelogram is then given by

$$Area = \det\begin{bmatrix} 2 & 1 \\ -4 & 5 \end{bmatrix} = 14$$

Section 4.1

Problem 2:

(1) Let $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ be a vector in W. Then, for any scalar $c \in \mathbb{R}$, we have $c\mathbf{u} = \begin{bmatrix} cx \\ cy \end{bmatrix}$. Notice that the product of the co-ordinates for $c\mathbf{u}$ is $(cx)(cy) = c^2xy$ which has the same sign as xy i.e $(cx)(cy) \geq 0$. This implies that $c\mathbf{u}$ is in W.

(2) Consider the vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. Then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The product of the co-ordinates for this vector is -1 which is less than 0. Hence it is not in W, which is the reason why W is not a subspace.

Problem 12: Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \quad , \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$

in $\mathbb{R}^4$. Then notice that W is the set of vectors of the form $s\mathbf{u} + t\mathbf{v}$ for any scalars s and t i.e vectors that are arbitrary linear combinations of $\mathbf{u}$ and $\mathbf{v}$. This shows that $W = Span(\mathbf{u}, \mathbf{v})$ and so by Theorem 1, it is a linear subspace of $\mathbb{R}^4$

Problem 20:

(1) To demonstrate that $C[a,b]$ is a subspace we need to show:
   (a) Scalar multiples of continuous functions are continuous i.e if $f \in C[a,b]$, then for any scalar $c \in \mathbb{R}$ its true that $cf \in C[a,b]$
   (b) Finite sums of continuous functions are continuous; in fact its enough to know that the sum of two continuous functions is continuous i.e if $f \in C[a,b]$ and $g \in C[a,b]$ then $(f+g) \in C[a,b]$. Once we know this for two functions, we can use induction to establish this for finite sums.

   Once we know the above 2 facts, we use them to check that $C[a,b]$ satisfies the three properties in the definition of a subspace.

(2) The zero function $\mathbf{0}$ satisfies the property $\mathbf{0}(a) = \mathbf{0}(b) = 0$. Suppose f and g are two functions in $C[a, b]$ which satisfy $f(a) = f(b)$ and $g(a) = g(b)$. Then $(f + g)(a) = f(a) + g(a) = f(b) + g(b) = (f + g)(b)$, satisfying the required property. Thus the set is closed under vector addition. Finally if $f \in C[a, b]$ satisfies $f(a) = f(b)$ then for any scalar c, its obvious that $(cf)(a) = cf(a) = cf(b) = (cf)b$. Thus the set is closed under scalar multiplication and we have shown that it is a subspace.

Section 4.2

Problem 5: Solving $A\mathbf{x} = 0$, we can choose x_2 and x_4 to be free parameters s and t respectively. The solution space which is $Nul(A)$ is the set of vectors $\mathbf{x}$ that satisfy $x_1 = 2s - 4t$ and $x_3 = 9t$ and $x_5 = 0$. Thus $Nul(A)$ is the subspace spanned by the vectors:

$$\begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 16: The given set of vectors can be written as:

$$b \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix}$$

So the matrix A for which the above set is $Col(A)$ is:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

Section 4.3

Problem 14: By Theorem 6, the pivot columns of A form the bases for $Col(A)$. From the row reduced form B, we see that the pivot columns of A are columns 1, 3 and 5. Hence the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix}$$

form the bases for $Col(A)$

To find the bases for $Nul(A)$ is equivalent to solving $B\mathbf{x} = 0$ and finding the bases there. Here for the solution set, we get that x_2 and x_4 are free parameters s and t

respectively and the solution set can be written as:

$$\mathbf{x} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix}$$

Thus the bases for $Nul(A)$ are the vectors:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix}$$

Problem 25: No, the set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is not a bases for H because $\mathbf{v_1}$ and $\mathbf{v_3}$ are not in H. This means that the span of the 3 vectors is strictly bigger than H. In fact, the vector $\mathbf{v_1} + \mathbf{v_3}$ which is in H, along with $\mathbf{v_2}$ forms a bases of H.