MA225 (T2) HOMEWORK 6 SOLUTIONS

Section 5.2

Problem 11: The characteristic polynomial of A is given by $p(t) = \det(A - tI)$. Using cofactor expansion we get

$$\begin{aligned} p(t) &= \det \begin{bmatrix} 4-t & 0 & 0 \\ 5 & 3-t & 2 \\ -2 & 0 & 2-t \end{bmatrix} \\ &= (4-t)(3-t)(2-t) \end{aligned}$$

Problem 18: The eigenspace of the eigenvalue $\lambda = 5$ is the solution space of the equation $(A - 5I)\mathbf{x} = 0$. For this to be two-dimensional the number of linearly independent rows of $(A - 5I)$.

$$A - 5I = \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Row 3 is a multiple of Row 4, so we just have to worry about Rows 1,2 and 4. Its easy to see that if $h \neq 6$, these rows are linearly independent making the solution set have dimension 1. When $h = 6$, row 2 is the sum of rows 1 and 4, and so rows 1 and row 4 are the 2 linearly independent rows and thus the eigenspace is two-dimensional.

Problem 20: Suppose A is a $n \times n$ matrix. Recalling the properties of determinants

$$\det(A^T - \lambda I) = \det(A - \lambda I)^T = \det(A - \lambda I)$$

So $\det(A^T - \lambda I) = 0$ is equivalent to $\det(A - \lambda I) = 0$ and hence A and A^T have the same eigenvalues.

Section 5.3

Problem 16: The eigenspace for the eigenvalue $\lambda = 2$ is the solution space of $(A - 2I)\mathbf{x} = 0$. This has bases

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

The eigenspace for the eigenvalue $\lambda = 1$ is the solution space of $(A - I)\mathbf{x} = 0$. This is a line spanned by

$$\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

Let P be the matrix with columns the bases vectors for each of the eigenspaces i.e

$$P = \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Then

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 26: It is possible that the matrix A is non-diagonalizable. This shall happen when the third eigenspace together with first and second does not span $\mathbb{R}^7$, which in turn is possible when it is one-dimensional. This would mean that the geometric multiplicity (i.e the dimension of the eigenspace) of one of the eigenvalues is not equal (in which case it is always strictly lesser than) to the algebraic multiplicity of that eigenvalue.

Section 6.1

Problem 18:

$$\mathbf{y} \cdot \mathbf{z} = -3 \cdot 1 + 7 \cdot -8 + 4 \cdot 15 + 0 \cdot -7 = 1 \neq 0$$

Hence the vectors are not orthogonal.

Problem 23:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= 2 \cdot -7 + -5 \cdot -4 + -1 \cdot 6 = 0 \\ \|\mathbf{u}\|^2 &= 2 \cdot 2 + -5 \cdot -5 + -1 \cdot -1 = 30 \\ \|\mathbf{v}\|^2 &= -7 \cdot -7 + -4 \cdot -4 + 6 \cdot 6 = 101 \end{aligned}$$

Finally

$$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = 131$$

Section 6.2

Problem 9: We see that

$$\begin{aligned} \mathbf{u_1} \cdot \mathbf{u_2} &= 1 \cdot -1 + 0 \cdot 4 + 1 \cdot 1 = 0 \\ \mathbf{u_2} \cdot \mathbf{u_3} &= -1 \cdot 2 + 4 \cdot 1 + 1 \cdot -2 = 0 \\ \mathbf{u_3} \cdot \mathbf{u_1} &= 2 \cdot 1 + 1 \cdot 0 + -2 \cdot 1 = 0 \end{aligned}$$

This shows that these vectors are mutually orthogonal. Since we now have 3 mutually orthogonal vectors in $\mathbb{R}^3$, they form a bases for $\mathbb{R}^3$.

$$\begin{aligned} \mathbf{x} \cdot \mathbf{u_1} &= 8 - 3 = 5 \\ \mathbf{x} \cdot \mathbf{u_2} &= -8 - 16 - 3 = -27 \\ \mathbf{x} \cdot \mathbf{u_3} &= 16 - 4 + 6 = 18 \end{aligned}$$

Finally

$$\mathbf{x} = \frac{(\mathbf{x}\cdot\mathbf{u_1})\mathbf{u_1}}{\|\mathbf{u_1}\|^2} + \frac{(\mathbf{x}\cdot\mathbf{u_2})\mathbf{u_2}}{\|\mathbf{u_2}\|^2} + \frac{(\mathbf{x}\cdot\mathbf{u_3})\mathbf{u_3}}{\|\mathbf{u_3}\|^2} = \frac{5}{2}\mathbf{u_1} - \frac{27}{18}\mathbf{u_2} + \frac{18}{9}\mathbf{u_3}$$

Problem 15: The distance d of $\mathbf{y}$ to the line through the origin and $\mathbf{u}$ is given by:

$$d = \|\mathbf{y} - \frac{(\mathbf{y}\cdot\mathbf{u})\mathbf{u}}{\|\mathbf{u}\|^2}\|$$

Computing the various required quantities

$$\begin{aligned} \mathbf{y}\cdot\mathbf{u} &= 30 \\ \|\mathbf{u}\| &= \sqrt{8^2+6^2} = 10 \end{aligned}$$

Thus we get

$$\begin{aligned} d &= \|\mathbf{y} - \frac{30\mathbf{u}}{100}\| \\ &= \|\mathbf{y} - \frac{3}{10}\mathbf{u}\| \\ &= \sqrt{(\frac{6}{10})^2 + (-\frac{8}{10})^2} \\ &= 1 \end{aligned}$$

Section 6.3

Problem 8:

$$\mathbf{u_1}\cdot\mathbf{u_2} = 1\cdot -1 + 1\cdot 3 + 1\cdot -2 = 0$$

Hence

$$\begin{aligned} proj_W(\mathbf{y}) &= \frac{\mathbf{y}\cdot\mathbf{u_1}}{\|\mathbf{u_1}\|^2}\mathbf{u_1} + \frac{\mathbf{y}\cdot\mathbf{u_2}}{\|\mathbf{u_2}\|^2}\mathbf{u_2} \\ &= \frac{6}{3}\mathbf{u_1} + \frac{7}{14}\mathbf{u_2} \\ &= \begin{bmatrix} \frac{3}{2} \\ \frac{7}{2} \\ 1 \end{bmatrix} \end{aligned}$$

So then

$$\mathbf{y} = \begin{bmatrix} \frac{3}{2} \\ \frac{7}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{5}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix}$$

where the first vector is in W and the second vector is orthogonal to W.

Problem 12: The closest point to $\mathbf{y}$ in the subspace W is the point $proj_W(\mathbf{y})$. The calculation is entirely similar to the previous problem.

Problem 17:

(1)

$$
\begin{aligned}
U^T U &= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2
\end{aligned}
$$

Similarly it is easily seen that

$$
UU^T = \begin{bmatrix} \frac{8}{9} & -\frac{2}{9} & \frac{2}{9} \\ -\frac{2}{9} & \frac{5}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{5}{9} \end{bmatrix}
$$

(2)

$$
\begin{aligned}
proj_W(\mathbf{y}) &= \frac{\mathbf{y} \cdot \mathbf{u_1}}{\|\mathbf{u_1}\|^2}\mathbf{u_1} + \frac{\mathbf{y} \cdot \mathbf{u_2}}{\|\mathbf{u_2}\|^2}\mathbf{u_2} \\
&= 6\mathbf{u_1} + 3\mathbf{u_2} \\
&= \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}
\end{aligned}
$$

On the other hand

$$
(UU^T)\mathbf{y} = \begin{bmatrix} \frac{8}{9} & -\frac{2}{9} & \frac{2}{9} \\ -\frac{2}{9} & \frac{5}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{5}{9} \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}
$$

Thus $proj_W(\mathbf{y}) = (UU^T)\mathbf{y}$