Absolute Convergence, Ratio & Root tests

Note Title 3/9/2009

Theorem: If $\sum_{k=1}^{\infty} |a_k|$ converges,

then $\sum_{k=1}^{\infty} a_k$ converges, too.

Proof: Note, $-|a_k| \leq a_k \leq |a_k|$

$$0 \leq a_k + |a_k| \leq 2|a_k|$$

$$0 \leq \underline{\sum_{k=1}^{\infty} a_k + |a_k|} \leq 2 \sum_{k=1}^{\infty} |a_k|$$ – converges (by assumption)

$$S_N = \sum_{k=1}^{N} a_k + |a_k|$$ – monotonic and bounded sequence

Recall: Monotonic, bounded sequence converges.

Therefore, S_N converges, and so does

$$\sum_{k=1}^{\infty} a_k + |a_k| .$$

Now, $\displaystyle\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k + |a_k| - |a_k| =$

$$= \underbrace{\sum_{k=1}^{\infty} a_k + |a_k|}_{\text{converges}} - \underbrace{\sum_{k=1}^{\infty} |a_k|}_{\text{converges}}$$

Hence, $\displaystyle\sum_{k=1}^{\infty} a_k$ converges, too. $\square$

This theorem justifies

Def 1: $\sum_{k=1}^{\infty} a_k$ is absolutely convergent if $\sum_{k=1}^{\infty} |a_k|$ converges.

Example 1 $\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k}$ converges absolutely since $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges.

geometric series

Example 2: $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ converges,

since $\sum_{k=1}^{\infty} \left|\frac{\sin(k)}{k^2}\right| \leq \sum_{k=1}^{\infty} \frac{1}{k^2}$ – converges by p-series test.

Comparison test

Example 3: $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ – converges by alternating series test.

On the other hand, $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

by the integral test.

So, this series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ is conditionally convergent

Ratio Test

$$\sum_{k=1}^{\infty} a_k, \quad a_k \neq 0 \text{ for all } k.$$

Suppose $\displaystyle\lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|} = L.$

(i) $L < 1$, the series converges absolutely

(ii) $L > 1$, " diverges

(iii) $L = 1$, no conclusion

Example 1: $\sum_{k=1}^{\infty} c \cdot r^k$, $a_k = c \cdot r^k$

$$\lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k \to \infty} \frac{|c \cdot r^{k+1}|}{|c r^k|} = |r| = L$$

According to the ratio test

If $|r| = L < 1$, the series converges absolutely

If $|r| = L > 1$, " diverges

Example 2: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{2^k}$ $\qquad \dfrac{2^k}{2^{k+1}} = \dfrac{1}{2}$

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{(k+1)^2}{2^{k+1}} \frac{2^k}{k^2} =$$

$$= \frac{1}{2} \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^2 = \frac{1}{2} \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^2 = \frac{1}{2} < 1$$

Hence, the series is absolutely convergent.

Example 3: $\sum_{k=1}^{\infty} \frac{1}{k}$,

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} = 1$$

Root Test: $\sum_{k=1}^{\infty} a_k$, $\lim_{k\to\infty} \sqrt[k]{|a_k|} = L$

(i) $L < 1$, absolute convergence

(ii) $L > 1$ $(= \infty)$, divergence

(iii) $L = 1$, no conclusion.

Example 1: $\sum_{k=1}^{\infty} c r^k$, $c \neq 0$

$$\lim_{k \to \infty} \sqrt[k]{|a_k|} = \lim_{k \to \infty} \sqrt[k]{|c| \cdot |r|^k} =$$

$$= |r| \lim_{k \to \infty} \sqrt[k]{|c|} = |r|.$$