Applications of Taylor Series

Note Title 3/18/2009

From the last lecture:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = P_n(x) + R_n(x)$$

remainder – $R_n(x) = \dfrac{e^z}{(n+1)!} x^{n+1}$, where z is between 0 and x.

$M = \max\{1, e^x\}$, then $|e^z| \le M$

Show that remainder converges to zero.

(Then, Taylor series of e^x converges to e^x)

$$|R_n(x)| = \frac{e^z}{(n+1)!}|x|^{n+1} \leq M \frac{|x|^{n+1}}{(n+1)!}$$

It is enough to show:

(*) $$\lim_{m\to\infty} \frac{|x|^m}{m!} = 0$$

To establish (*), let $N > |x|$, $m > N$.

Then $\dfrac{|x|^m}{m!} \leq \dfrac{N^m}{m!} = \dfrac{\overbrace{N \cdot N \cdots N \cdots N}^{m}}{\underbrace{1 \cdot 2 \cdots N \cdots m}_{m}} \leq$

$\leq N^N \cdot \dfrac{N}{m} = \dfrac{N^{N+1}}{m} \to 0$

as $m \to \infty$.

Therefore, $\lim\limits_{m \to \infty} \dfrac{|x|^m}{m!} = 0$.

Maclaurin Series is Taylor Series centered at $x=0$, i.e. $c=0$.

1. Find new Taylor Series from the known ones.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{4k+2}$$

2. Using Taylor Series to evaluate limits.

$$\lim_{x\to 0} \frac{\sin(x^3) - x^3}{x^9} = ?$$

$$\sin(x^3) = x^3 - \frac{(x^3)^3}{3!} + \frac{(x^3)^5}{5!} + \ldots =$$

$$= x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} + \ldots$$

$$\lim_{x\to 0} \frac{\sin(x^3) - x^3}{x^9} = \lim_{x\to 0} \frac{\cancel{x^3} - \frac{x^9}{6} + \frac{x^{15}}{5!} - \ldots - \cancel{x^3}}{x^9} =$$

$$= \lim_{x \to 0} \left(-\frac{1}{6} + \frac{x^6}{5!} - \cdots \right) = -\frac{1}{6}.$$

3. Using Taylor Series to approximate integrals

$$\int_0^1 \frac{\sin x}{x}\, dx$$ (This is improper integral but we can treat it as a proper one since $\lim_{x \to 0} \frac{\sin x}{x} = 1$)

$$\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x}\,dx =$$

$$= \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right) dx =$$

$$= \left[x - \frac{x^3}{3!\,3} + \frac{x^5}{5!\,5} - \dots\right]_0^1 \approx 1 - \frac{1}{3!\,3} + \frac{1}{5!\,5}$$