Binomial Series

Note Title 4/6/2009

Recall binomial theorem in Algebra

$$(a+b)^n = a^n + n a^{n-1} b + n(n-1) a^{n-2} b^2 + \dots + n a b^{n-1} + b^n$$

n – positive integer

$\binom{n}{k}$ – binomial coefficient

$$\binom{n}{0} = 1, \quad \binom{n}{1} = n, \quad \binom{n}{2} = n(n-1)$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{n(n-1)\ldots(n-k+1)}{k!}, \quad k \geq 3$$

Let $a = 1$, $b = x$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

(also it is Taylor series of $(1+x)^n$, centered at $x = 0$.

Newton extended this formula for general n, including negative and non-integer values.

Now, assume $n \neq 0$. Find Taylor series of $f(x) = (1+x)^n$, centered at "0".

not a positive integer

$f(x) = (1+x)^n$ | $f(0) = 1$

$f'(x) = n(1+x)^{n-1}$ | $f'(0) = n$

$$f''(x) = n(n-1)(1+x)^{n-2} \qquad f''(0) = n(n-1)$$

$$\cdots\cdots \qquad \cdots$$

$$f^{(k)}(x) = n(n-1)\ldots(n-k+1)(1+x)^{n-k} \qquad f^{(k)}(0) = n(n-1)\ldots(n-k+1)$$

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

if the series converges and the remainder goes to "0".

Question: Convergence

Use ratio test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\binom{n}{k+1}x^{k+1}}{\binom{n}{k}x^k}\right| =$$

$$= |x|\lim_{k\to\infty}\left|\frac{n(n-1)\ldots(n-k)}{(k+1)!}\cdot\frac{k!}{n(n-1)\ldots(n-k+1)}\right| =$$

$$|x|\lim_{k\to\infty}\frac{|n-k|}{k+1} = |x|\cdot 1 = |x|$$

$$\lim_{k\to\infty} \frac{|n-k|}{k+1} = \lim_{k\to\infty} \frac{\left|\frac{n}{k} - 1\right|}{1+\frac{1}{k}} = \frac{|0-1|}{1} = 1$$

If $|x| < 1$, then binomial series converges absolutely and diverges if $|x| > 1$.

It can be shown that the remainder

$\lim_{n\to\infty} R_n(x) = 0$ if $|x| < 1$. $\boxed{\text{Thm 8.1}}$

Example: $\sqrt{1+x} = (1+x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{n}{k} x^k$

By def:

$k=0$ $\qquad$ $k=1$

$$\binom{n}{0} = 1, \quad \binom{n}{1} = n$$

$$\binom{1/2}{0} = 1, \quad \binom{1/2}{1} = \frac{1}{2}$$

$k=2$

$$\binom{1/2}{2} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n(n-1)}{2} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2} = -\frac{1}{8}$$

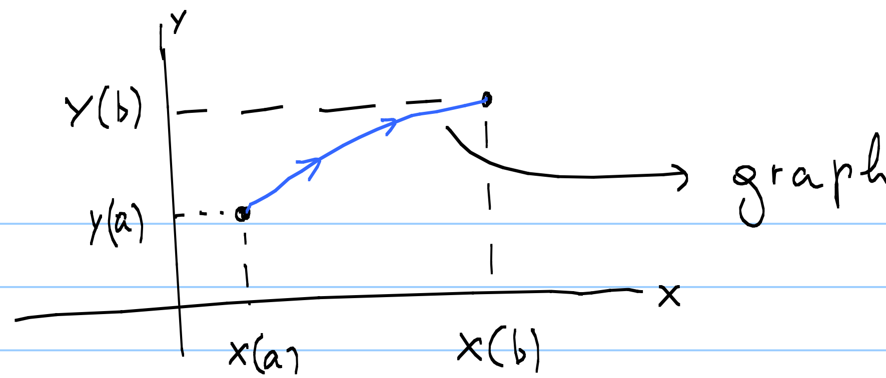
$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$$

Chapter 9

Plane curves and parametric equations.

$x = x(t)$
$y = y(t)$ parametric equations of a curve

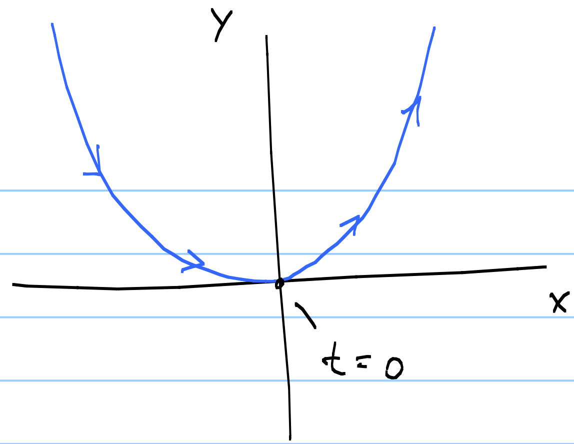
$a \le t \le b$



Example

$x(t) = t \qquad -\infty < t < +\infty$

$y(t) = t^2$



$t = 0 \rightarrow (x, y) = (0, 0)$

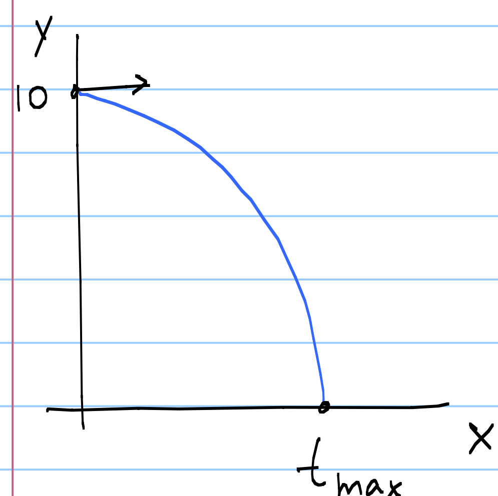
$t = 1 \rightarrow (x, y) = (1, 1)$

Note: $x(t) = t \rightarrow x^2(t) = t^2$

$y(t) = t^2 \quad y(t) = t^2 \Rightarrow x^2 = y$

the same curve

Example: Path of a projectile



$$x = t$$

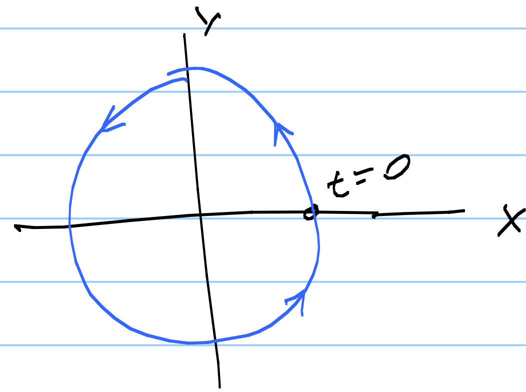
$$y = 10 - t^2$$

$$0 \le t \le t_{max}$$

$$0 = y(t_{max}) = 10 - t_{max}^2 = 0$$

$$\Rightarrow t_{max} = \sqrt{10}$$

Example $x(t) = \cos t$
$y(t) = \sin t$, $0 \le t \le 2\pi$



$x^2(t) = \cos^2 t$
$y^2(t) = \sin^2 t$

$x^2 + y^2 = 1$

circle of radius 1.