Calculus and polar coordinates

Note Title 4/29/2009

Quiz tomorrow!

Find a tangent line to a curve in polar coordinates:

$$r = f(\theta)$$

(*)
$$\left.\frac{dy}{dx}\right|_{\theta=a} = \frac{\frac{dy}{d\theta}(a)}{\frac{dx}{d\theta}(a)}$$

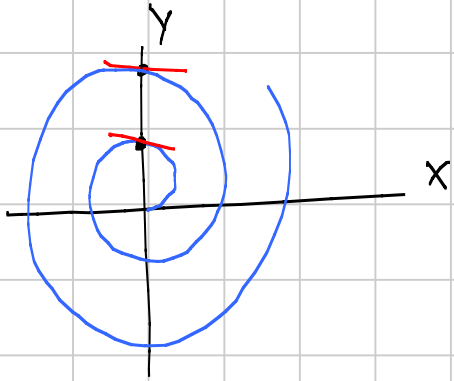
(By calculus of parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

$$\left.\begin{aligned}\frac{dx}{d\theta} &= f'(\theta)\cos\theta - f(\theta)\sin\theta \\ \frac{dy}{d\theta} &= f'(\theta)\sin\theta + f(\theta)\cos\theta\end{aligned}\right\}\ \text{substit. in } (*)$$

$$(*)\rightarrow \quad \left.\frac{dy}{dx}\right|_{\theta=a} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

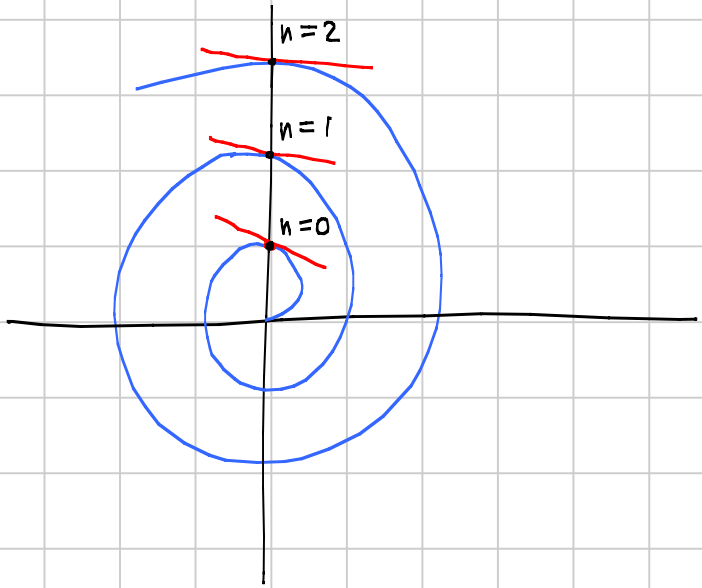
Example Archimedian Spiral $r = \theta$, $\theta \geq 0$



Find the slope at $\theta = \frac{\pi}{2} + 2n\pi$, $n = 0, 1, 2, \ldots$

$$\left.\frac{dy}{dx}\right|_{\theta = \frac{\pi}{2} + 2n\pi} = \frac{1 \cdot \sin\theta + \theta \cos\theta}{1 \cdot \cos\theta - \theta \sin\theta}\Bigg|_{\theta = \frac{\pi}{2} + 2n\pi} =$$

$$= \frac{1 + \left(\frac{\pi}{2} + 2n\pi\right) \cdot 0}{1 \cdot 0 - \left(\frac{\pi}{2} + 2n\pi\right) \cdot 1} = -\frac{1}{\frac{\pi}{2} + 2n\pi}$$



$n = 0, \quad -\frac{2}{\pi}$ - slope

$n = 1, \quad -\frac{1}{\frac{\pi}{2} + 2\pi}$

$n = 2, \quad -\frac{1}{\frac{\pi}{2} + 4\pi}$

Q: At which θ, the tangent is horizontal?

$$\left.\frac{dy}{dx}\right|_{\theta=a} = \frac{\sin a + a\cos a}{\cos a - a\sin a} = 0 \iff$$

$$\sin a + a\cos a = 0$$
$$\cos a - a\sin a \neq 0 \implies$$

(1) $\tan a + a = 0$

(2) $\cot a - a \neq 0$

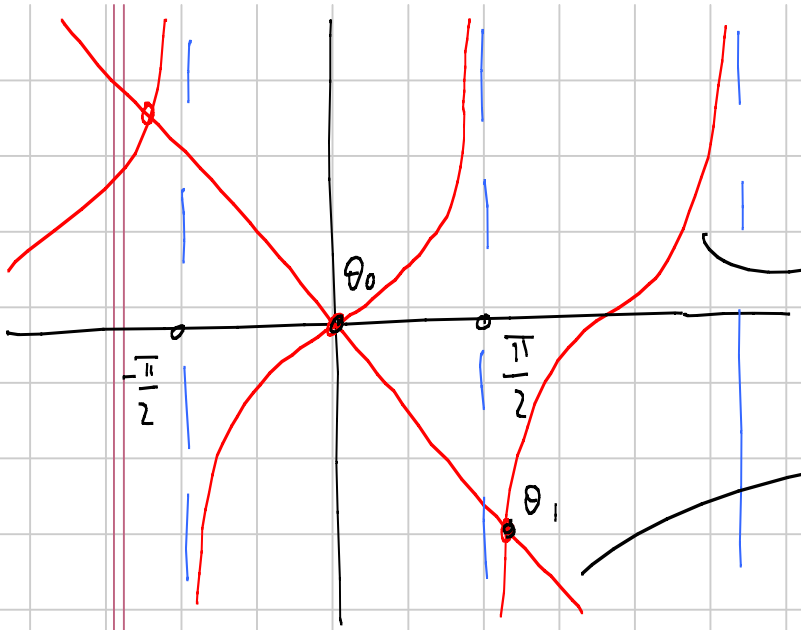
If (1) is true then $\cot a - a =$

$= \cot a + \tan a = \frac{\cos a}{\sin a} + \frac{\sin a}{\cos a} =$

$= \frac{1}{\sin a \cdot \cos a} \neq 0$

Only need to find roots of $\tan a + a = 0$

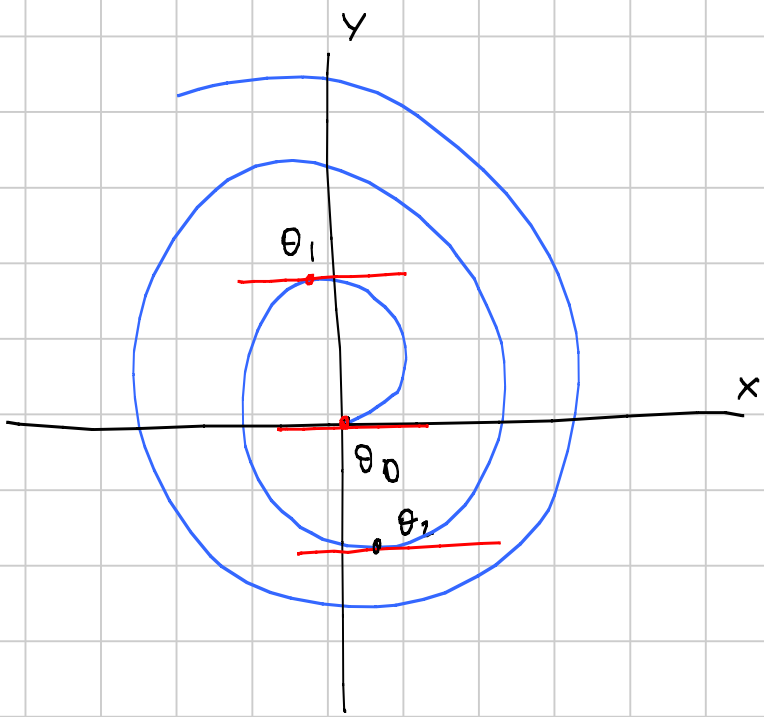
Solve graphically.



$\tan a + a = 0$

$y = \tan a$

$y = -a$

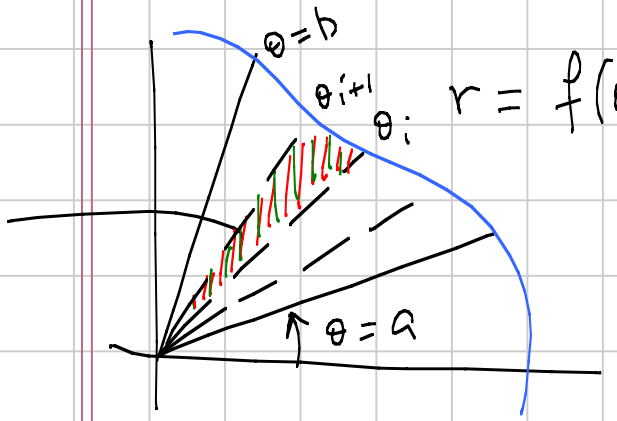


$r = \theta, \quad \theta \geq 0$

$\theta_0 = 0$

$\theta_1 = \frac{\pi}{2} + \text{small}$

Area in Polar coordinates



A_i = area

Divide $[a,b]$ into n subintervals:

$$a = \theta_0 < \theta_1 < \dots < \theta_n = b$$

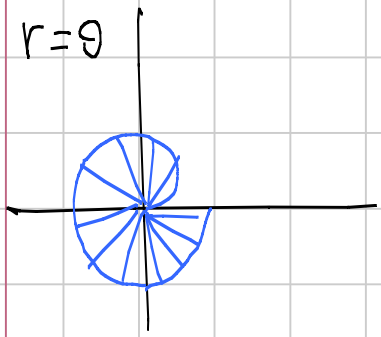
$$\Delta\theta = \theta_{i+1} - \theta_i = \frac{b-a}{n}$$

$$A_i = \text{area of red} \approx \frac{1}{2} r_i^2 \Delta\theta = \frac{1}{2} f(\theta_i)^2 \Delta\theta$$

total area $\sim$ $A \approx \sum_{i=0}^{n-1} A_i = \frac{1}{2} \sum_{i=0}^{n-1} f^2(\theta_i)\,\Delta\theta$

$$A = \lim_{n\to\infty} \sum_{i=0}^{n-1} A_i = \frac{1}{2}\int_a^b f^2(\theta)\,d\theta$$

Example Area enclosed by a spiral arc from $\theta = 0$ to 2π



$$A = \frac{1}{2}\int_0^{2\pi} \theta^2 \, d\theta = \frac{1}{2}\left[\frac{\theta^3}{3}\right]_0^{2\pi} =$$

$$= \frac{1}{2}\,\frac{(2\pi)^3}{3} = \frac{4}{3}\pi^3 .$$