Comparison Tests

Note Title 3/4/2009

Comparison Test (similar to improper integrals)

If $0 \le a_k \le b_k$, for all k.

$$\sum_{k=1}^{\infty} a_k \le \sum_{k=1}^{\infty} b_k$$

If $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ converges, too.

If $\sum_{k=1}^{\infty} a_k$ diverges then $\sum_{k=1}^{\infty} b_k$ diverges, too.

Example 1 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+k^2+1} \leq \sum_{k=1}^{\infty} \frac{1}{k^3}$ — converges by p-series test $(p=3)$

$k^3+k^2+1 \geq k^3 \rightarrow \dfrac{1}{k^3+k^2+1} \leq \dfrac{1}{k^3}$

Recall p- series test :

$p>0$ $$\sum_{k=1}^{\infty} \frac{1}{k^p} - \begin{cases} \text{converges} & \text{if } p>1 \\ \text{diverges} & \text{if } p \le 1 \end{cases}$$

Limit Comparison Test

Suppose $a_k, b_k \geq 0$ and $\lim_{k \to \infty} \frac{a_k}{b_k} = L > 0$

Then, $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ either both converge or diverge.

Example 2 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+k^2+1} = \sum_{k=1}^{\infty} a_k$

Let $b_k = \dfrac{1}{k^3}$, $\displaystyle\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^3}{k^3+k^2+1} =$

$$= \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}+\frac{1}{k^3}} = 1.$$

By p-series test $\displaystyle\sum_{k=1}^{\infty} b_k$ converges, so

by limit comparison test $\sum a_k$ converges, too.

Alternating Series

$a_k > 0$ $\quad \displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \dots$

Example 1 Alternating harmonic series

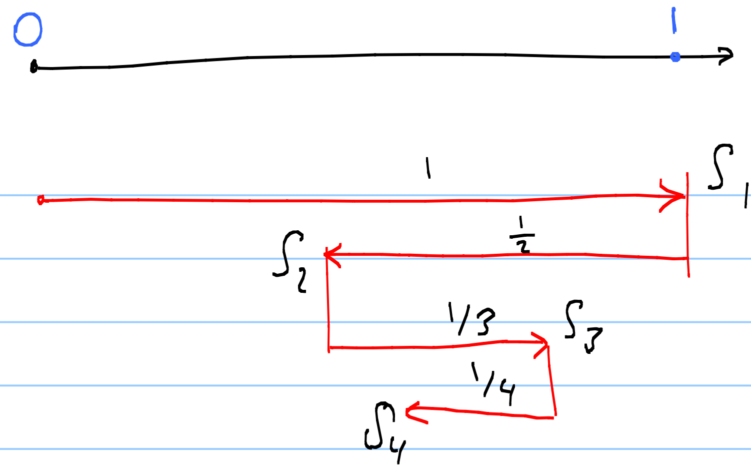
$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Theorem Suppose $\lim_{k \to \infty} a_k = 0$,

$$0 < a_{k+1} < a_k .$$

Then $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$



$$S_2 < S_4 < S_6 < \dots < S_{2n} < S_{2k+1} < \dots < S_5 < S_3 < S_1$$

$\{S_2, S_4, S_6, \dots\}$ is bounded (by S_1) and increasing

$\{S_1, S_3, S_5, \dots\}$ is bounded (by "0") and decreasing

Recall: monotonic bounded sequence converges.

$$\lim_{n\to\infty} S_{2n} = L_1 \quad , \quad \lim_{k\to\infty} S_{2k+1} = L_2$$

But $\quad S_{2n+1} - S_{2n} = a_{2n+1}$

$$\underbrace{\lim_{n\to\infty} S_{2n+1}}_{L_1} - \underbrace{\lim_{n\to\infty} S_{2n}}_{L_2} = \lim_{n\to\infty} a_{2n+1} = 0$$

$$L_1 - L_2 = 0$$

Hence, $\lim_{n\to\infty} S_n = L = L_1 = L_2$, so the series converges.

End of proof

Example: $\sum_{k=1}^{\infty} (-1)^k \frac{k+3}{k(k+1)}$

$$\lim_{k\to\infty} \frac{k+3}{k(k+1)} = 0, \qquad \frac{a_{k+1}}{a_k} =$$

$$= \frac{k+4}{\cancel{(k+1)}(k+2)} \cdot \frac{k\cancel{(k+1)}}{k+3} = \frac{k(k+4)}{(k+2)(k+3)} = \frac{k^2+4k}{k^2+5k+6} <$$

$$< 1 \text{ if } k \geq 1, \text{ i.e. } a_k > a_{k+1} > 0$$

By the Theorem, this series converges.