Exponential growth and decay problems

Note Title 2/11/2009

differential equations

Observation: The rate of growth of bacterial culture is proportional to the current population.

Let $y(t)$ = # of bacteria at time t.

The rate of change is $\Delta y \approx y \, \Delta t$

$$\frac{\Delta y}{\Delta t} = y'(t) \quad \text{as } \Delta t \to 0$$

Then, $y'(t)$ is proportional to $y(t)$

(*) $$\boxed{y'(t) = k\,y(t)}, \quad k \text{ - constant}$$

This is an example of differential equation

— it involves unknown function and its derivative.

Restriction on $y(t)$: $y(t) \geq 0$

Goal: Find $y(t)$ such that (*) holds.

$$y'(t) = k\,y(t)$$

$$\frac{y'(t)}{y(t)} = k \qquad \int \frac{y'(t)}{y(t)}\,dt = \int k\,dt$$

For $\int \frac{y'(t)}{y(t)}\,dt$, use substitution $t \to y$

$$\int \frac{1}{y(t)} \underbrace{y'(t)\,dt}_{dy} = \int \frac{1}{y}\,dy = \ln|y| + C_1$$

$$\int k\,dt = kt + C_2$$

$$\ln|y| + C_1 = kt + C_2$$

$$\ln|y| = kt + C, \quad C = C_2 - C_1$$

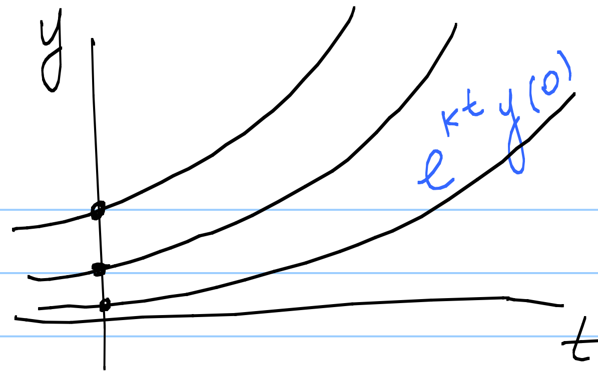
$$|y| = e^{kt} e^{C} = e^{kt} D, \quad D = e^{C}$$

Since $y \geq 0$,

$y = e^{kt} D$, let $t = 0$, then

$y = D$ – population at $t = 0$.

$$y(t) = e^{kt} y(0)$$



exponential growth

Radioactive Decay

$0 < y(t)$ - mass of radioactive material

$$y'(t) = k\,y(t), \quad k < 0$$

^{14}C - carbon-14, half-life $= 5730$ years

Problem: 50 grams today $(t=0)$

Question: How much will be left in 100 years?

$$y(t) = e^{kt} y(0), \quad y(0) = 50$$

Need to find k:

$$\frac{50}{2} = e^{k\cdot(\text{half-life})} 50$$

$$25 = e^{k\cdot 5730} 50$$

$$e^{k\cdot 5730} = \frac{1}{2} \qquad k = \frac{\ln\frac{1}{2}}{5730} = \frac{-\ln 2}{5730}$$

$$k \approx -1.2\cdot 10^{-4}$$

$$y(100) = e^{-1.2\cdot 10^{-4}\cdot 100} 50 = e^{-0.012}\cdot 50$$

Newton's law of cooling

$y(t)$ - temperature of water

T_a - ambient temperature (e.g. air)

$$y'(t) = k\,(y(t) - T_a)$$

Solve the equation

$$\frac{y'(t)}{y(t) - T_a} = k \qquad \text{integrate}$$

$$\int \frac{y'(t)}{y(t) - T_a}\, dt = \int k \, dt$$

$$dy = y'(t)\, dt$$

$$\int \frac{1}{y - T_a}\, dy = kt + C_2$$

$$\ln|y - T_a| + C_1 = kt + C_2$$

$$\ln|y - T_a| = kt + C, \quad C = C_2 - C_1$$

$$|y - T_a| = e^{kt} D, \quad D = e^{C}$$

Assume $y > T_a$

$$y - T_a = e^{kt} D$$

$$y(t) = T_a + e^{kt} D$$

At $t = 0$, $y(0) = T_a + D \Rightarrow D = y(0) - T_a$

$$y(t) = T_a + e^{kt}(y(0) - T_a)$$

$$y(t) = e^{kt} y(0) + T_a(1 - e^{kt})$$