Fourier Series

Note Title 3/30/2009

No quiz this Thursday

Periodic function:

$f(x)$ is T-periodic if for some $T > 0$, $f(x+T) = f(x)$ for any x.

Example

$f(x) = C$ - constant

$f(x) = \sin(x)$, $f(x) = \cos(x)$

$f(x) = \sin(kx)$, k is an integer

$f(x) = \cos(kx)$, k is an integer

All these functions are 2π-periodic

e.g. $\underline{f(x+2\pi)} = \sin(k(x+2\pi)) = \sin(kx + 2\pi k) =$

$= \sin(kx) = \underline{f(x)}$

Goal: Represent a 2π-periodic function $f(x)$ as a series of $\sin(kx)$ and $\cos(kx)$.

$$(*) \quad f(x) = \underbrace{\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx) \right)}_{\text{Fourier Series}}$$

$$\left.\begin{array}{l} a_0, a_1, a_2, \ldots, a_k, \ldots \\ b_1, b_2, \ldots \end{array}\right\}$$ Fourier coefficients

Note: Similar to Taylor Series but instead of power series, we use trigonometric series.

Q: Assuming that Fourier Series converges to $f(x)$, what are $\{a_k, b_k\}$?

Useful trig. identities:

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0, \text{ for any integer } m, n.$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \end{cases}$$

Now, integrate $(*)$

$$= \frac{a_0}{2} \cdot 2\pi = a_0 \pi$$

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx +$$

$$= 0$$

$$+ \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} a_k \cos(kx)\,dx + \int_{-\pi}^{\pi} b_k \sin(kx)\,dx$$

$$= a_k \int_{-\pi}^{\pi} \cos(kx)\,dx = \frac{a_k}{k} \sin(kx) \Big|_{-\pi}^{\pi} = 0$$

So, $\quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$

Multiply $(*)$ by $\cos(nx)$ and integrate

$n \neq 0, n > 0$

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\cos(nx)\,dx \;+$$

$$+ \sum_{k=1}^{\infty} a_k \int_{-\pi}^{\pi} \cos(kx)\cos(nx)\,dx + b_k \overbrace{\int_{-\pi}^{\pi} \sin(kx)\cos(nx)\,dx}^{=0}$$

$$= a_n \int_{-\pi}^{\pi} \cos^2(nx)\,dx = a_n \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \quad , \quad n = 1, 2, \ldots$$

Now, by multiplying (*) with $\sin(nx)$ and integrating we similarly obtain

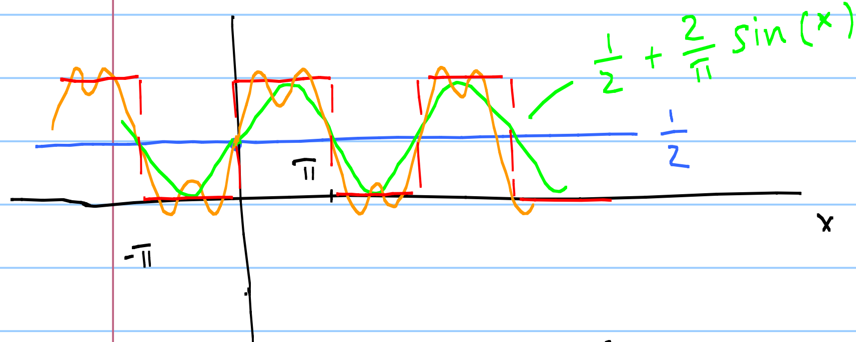
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx .$$

Example

$$f(x+2\pi) = f(x)$$

$$f(x) = 0 \quad \text{if} \quad -\pi < x \le 0$$

$$f(x) = 1 \quad \text{if} \quad 0 < x \le \pi$$



Find Fourier Series

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1 \cdot dx = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{0}^{\pi} \cos(nx)\,dx =$$

$$= \frac{1}{\pi \cdot n} \sin(nx)\Big|_{0}^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)\,dx =$$

$$\frac{-1}{\pi n}\cos(nx)\Big|_0^{\pi} = \frac{1}{\pi \cdot n}\Big(1-\cos(n\pi)\Big) =$$

$$= \begin{cases} 0 & \text{if} \quad n \text{ is even} \\ \dfrac{2}{\pi n} & \text{if} \quad n \text{ is odd.} \end{cases}$$

$$f(x) = \frac{1}{2} + \sum_{n-\text{odd}} \frac{2}{\pi n}\sin(nx) = \frac{1}{2} + \frac{2}{\pi}\sin(x) + \frac{2}{3\pi}\sin(3x) + \dots$$