Improper Integrals

Note Title 2/4/2009

<u>Example 1</u> (Wrong calculation)

$$\int_{-2}^{2} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-2}^{2} = -\frac{1}{2} - \left(-\frac{1}{-2}\right) = -1$$

This cannot be true since $\frac{1}{x^2} \geq 0$

on $-2 \leq x \leq 2$, but integral of

positive function must be positive.

What is wrong? The fundamental theorem of Calculus requires integrand to be continuous on the integration interval.

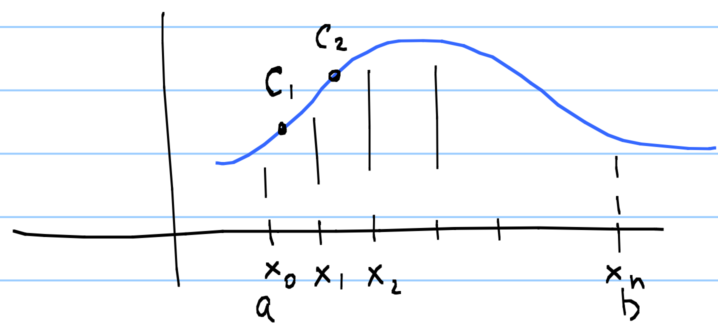
But $\frac{1}{x^2}$ is not continuous at $x=0$.

Recall the definition

must be the same for any choice of c_i

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

$$x_{i-1} \le c_i \le x_i$$



But if $f(c_i) = \infty$, then the limit is not defined!

In our example $\left.\frac{1}{x^2}\right|_{x=0} = \infty$.

Such integrals are called <u>improper</u>.

Need special definition.

<u>Example 2</u>: $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$

Note, $\left.\frac{1}{\sqrt{1-x}}\right|_{x=1} = \infty$

<u>Recall</u>: Definite integral = Area under the curve

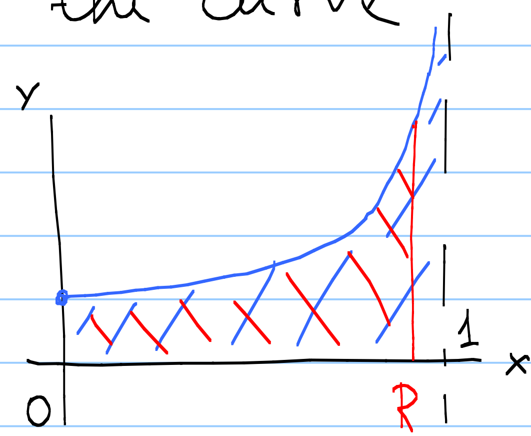
Approximate blue area by the red one:

$$\int_0^R \frac{1}{\sqrt{1-x}}\,dx = \Big[-2\sqrt{1-x}\Big]_0^R =$$

$$= -2\sqrt{1-R} + 2\sqrt{1-0} =$$

$$= 2 - 2\sqrt{1-R}$$

this is a proper integral



$0 < R < 1$

$$\lim_{\substack{R\to 1\\ R<1}} \int_0^R \frac{1}{\sqrt{1-x}}\,dx = \lim_{\substack{R\to 1\\ R<1}} \left(2-2\sqrt{1-R}\right) = 2-2\sqrt{1-1} = 2$$

Definition (6.1): If $f(x)$ is continuous on $[a,b)$ and $|f(x)| \to \infty$ as $x \to b^-$ $\left(\begin{matrix} x\to b \\ x<b \end{matrix}\right)$

then $$\int_a^b f(x)\,dx = \lim_{R\to b^-} \int_a^R f(x)\,dx$$

If the limit exists, improper integral converges

If the limit does not exist . " " diverges.

Similar definition for $\int_a^b f(x)\,dx$ if $|f(x)| \to \infty$ as $x \to a^+$.

(See the text)

Example 3. $\int_0^1 \frac{1}{1-x}\,dx = \lim_{R\to 1^-} \int_0^R \frac{1}{1-x}\,dx =$

$= \lim_{R\to 1^-} \Big[-\ln|1-x|\Big]_0^R = \lim_{R\to 1^-} \Big[-\ln|1-R| + \underset{\overset{\shortparallel}{0}}{\ln 1}\Big]$

$= \lim_{R\to 1^-} \left(-\ln(1-R)\right) = \infty$ — limit

does not exist $\Longrightarrow$

This improper integral diverges