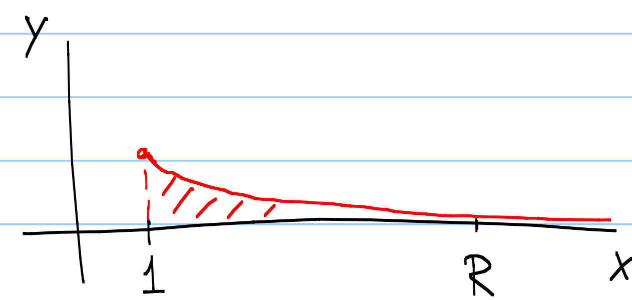
Improper Integrals with infinite limits of integration

Note Title 2/9/2009

$$\int_a^\infty f(x)\,dx := \text{Area under the curve}$$

Example: $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$



Consider $\displaystyle\int_1^R \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^R = 1 - \frac{1}{R}$

$$\lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx = \lim_{R\to\infty}\left(1-\frac{1}{R}\right) = 1$$

<u>Definition (6.3):</u> If $f(x)$ is continuous on $[a,\infty)$

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx .$$

If the limit exists $(=L)$, then

the improper integral converges $(=L)$.

If the limit does not exist then

the improper integral diverges. /

$$\int_1^{\infty} \frac{1}{x^2}\,dx = 1$$

Criterion of convergence:

If improper integral $\int_a^\infty f(x)\,dx$ converges,
then $f(x) \to 0$ as $x \to \infty$.

Example $\int_1^\infty \sin(x)\,dx$ diverges since $\sin(x)$ does not converge to zero as $x \to \infty$.

- The reverse need not be true

<u>Example</u> $\int_1^\infty \frac{1}{x}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx =$

$= \lim_{R\to\infty} \ln|x|\Big|_1^R = \lim_{R\to\infty} (\ln R - \underbrace{\ln 1}_{=0}) =$

$= \lim_{R\to\infty} \ln R = \infty$, $\Rightarrow$ integral diverges

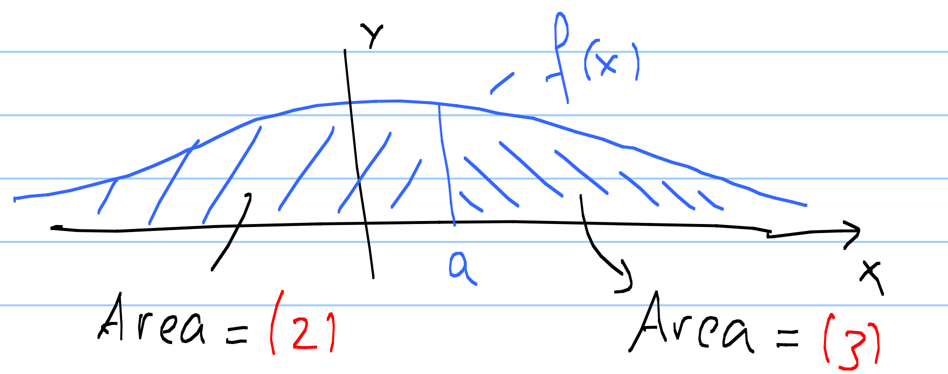
Def (6.4) If f is continuous on $(-\infty, \infty)$

$$\underbrace{\int_{-\infty}^{+\infty} f(x)\,dx}_{(1)} = \underbrace{\int_{-\infty}^{a} f(x)\,dx}_{(2)} + \underbrace{\int_{a}^{+\infty} f(x)\,dx}_{(3)},$$

for any a.

Integral (1) converges if both (2) and (3) converge.

(1) diverges if (2) or (3) diverges.

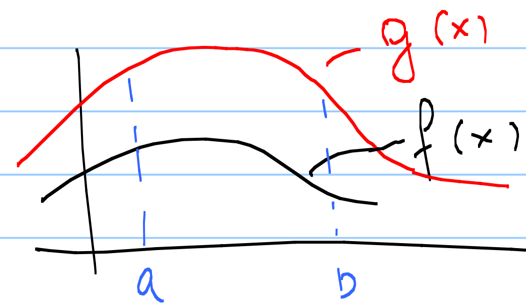


Comparison Test

Suppose $0 \le f(x) \le g(x)$, on $[a, b]$

then

$$0 \le \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$



Comparison Test: f, g are continuous for $a \leq x < \infty$ and $0 \leq f(x) \leq g(x)$ on $[a, \infty)$, then

If $\int_a^\infty g(x)\,dx$ converges then $\int_a^\infty f(x)\,dx$ also converges.

If $\int_a^\infty f(x)\,dx$ diverges then $\int_a^\infty g(x)\,dx$ also diverges.

Rk1: This test works for the other improper integrals.

Example: $\int_0^\infty \frac{1}{x^2+e^x}\,dx$, $\frac{1}{x^2+e^x}$ – continuous

$$\frac{1}{x^2+e^x} = f(x) \le \frac{1}{e^x} = g(x) \quad \text{on } [0,\infty)$$

$$\int_0^\infty \frac{1}{e^x}\,dx = \lim_{R\to\infty}\int_0^R \frac{1}{e^x}\,dx = \lim_{R\to\infty} -e^{-x}\Big|_0^R =$$

$$= \lim_{R\to\infty}\left(-e^{-R} + \underbrace{e^{-0}}_{1}\right) = 1$$

$\int_0^\infty \frac{1}{e^x}\,dx$ converges $\Rightarrow$ (By comparison test)

$\int_0^\infty \frac{1}{x^2+e^x}\,dx$ converges, too.