Infinite Series

Note Title 2/25/2009

Midterm 2 will be 4/1.

Quiz tomorrow: definitions of sequences and convergence.

Definition

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \ldots + a_n + \ldots$$

infinite series

Partial Sum $\quad S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$

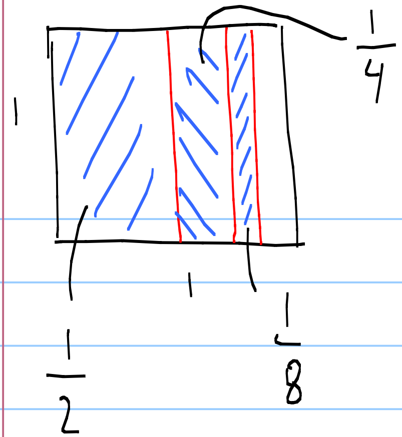
Partial Sums form a sequence $\quad \{S_n\}_{n=1}^{\infty}$

$$\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = S$$

If this limit exists, we say the series converges to S, called the sum of the series.

Otherwise, the series diverges.

<u>Example 1:</u> $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots = \sum_{k=1}^{\infty} \frac{1}{2^k}$



It turns out:

$$\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n} = S_n$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1$$

<u>Geometric Series</u> = series given by

$$\sum_{k=0}^{\infty} a\cdot r^k = \frac{a}{1-r} \quad \text{if } |r| < 1.$$

If $|r| \geq 1$ and $a \neq 0$, then the geometric series diverges.

Proof ($|r| < 1$)

$$S_n = ar^0 + ar^1 + \dots + a \cdot r^{n-1}$$

$$S_n \cdot r = ar^1 + ar^2 + \dots + ar^n$$

$$S_n \cdot r - S_n = ar^n - ar^0$$

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$\sum_{k=0}^{\infty} a r^k = \lim_{n \to \infty} S_n = \lim_{n \to \infty} a \frac{r^n - 1}{r-1} =$$

$$= a \frac{\lim_{n \to \infty} \overset{=0}{r^n} - 1}{r-1} = a \frac{-1}{r-1} = a \frac{1}{1-r}$$

End of proof

$$\left(\lim_{n \to \infty} r^n = 0, \quad \text{since} \quad |r| < 1 \right)$$

Example 2: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \sum_{k=1}^{\infty} \frac{1}{2^k} =$

$= \sum_{k=0}^{\infty} \frac{1}{2} \left(\frac{1}{2} \right)^k$ → Apply our formula for the sum of geometric series

with $\boxed{a = \frac{1}{2}}$, $\boxed{r = \frac{1}{2}}$ $|r| < 1$

$$S = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

Example 3

$$5 + \frac{5}{10} + \frac{5}{100} + \ldots$$

$$a = 5, \quad r = \frac{1}{10}$$

$$\sum_{k=0}^{\infty} 5\left(\frac{1}{10}\right)^k = \frac{5}{1-\frac{1}{10}} = \frac{50}{9}$$

Theorem If $\sum_{k=1}^{\infty} a_k$ converges,

then $\lim_{n\to\infty} a_n = 0$

Proof: If the series converges

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = S$$

We also have $S_n = \sum_{k=1}^{n} a_k = \overbrace{\left(\sum_{k=1}^{n-1} a_k\right)}^{= S_{n-1}} + a_n =$

$$= S_{n-1} + a_n$$

$$a_n = S_n - S_{n-1}$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (S_n - S_{n-1}) = S - S = 0$$

So, $\lim_{n \to \infty} a_n = 0$.

End of proof

The theorem implies

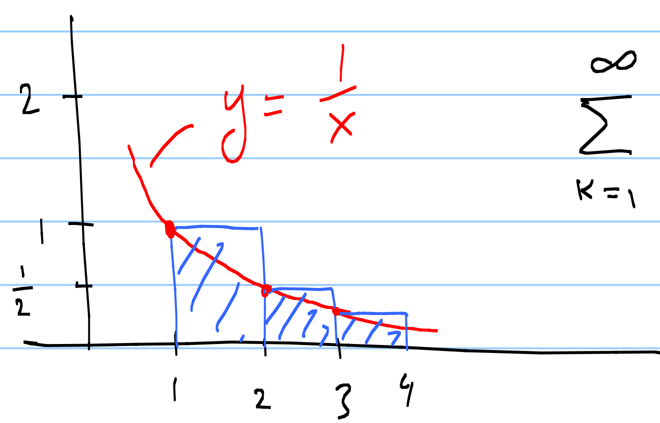
k-th term test

If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Example 4 If $\lim_{n\to\infty} a_n = 0$, the series may diverge.

Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k} \text{ — diverges}$$



$$\sum_{k=1}^{\infty} \frac{1}{k} \geq \int_{1}^{\infty} \frac{1}{x}\,dx = \infty$$