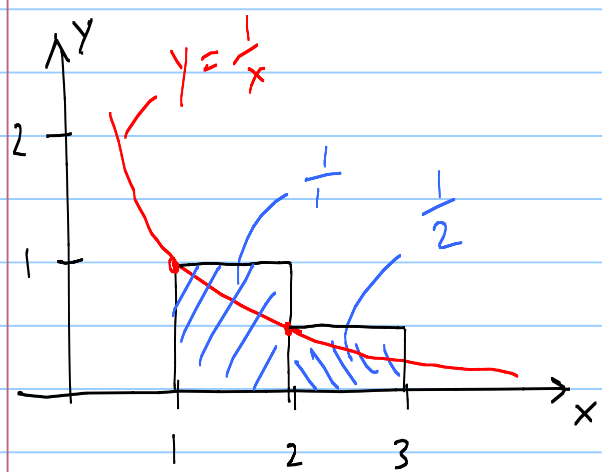
Integral Test

Note Title 3/2/2009

Recall Harmonic Series: $\sum_{k=1}^{\infty} \frac{1}{k}$ - Convergence?



$$\frac{1}{1} \geq \int_1^2 \frac{1}{x}\,dx$$

$$\frac{1}{2} \geq \int_2^3 \frac{1}{x}\,dx$$

. . . .

$$\frac{1}{n} \geq \int_{n}^{n+1} \frac{1}{x}\,dx$$

$$S_n = \sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n+1} \frac{1}{x}\,dx = \ln(n+1)$$

$$" \left[\ln|x|\right]_1^{n+1} " = \ln|n+1| - \ln|1|$$

$$\ln|1| " = " 0$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = \lim_{n\to\infty} S_n \geq \lim_{n\to\infty} \ln(n+1) = \infty$$

Harmonic Series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

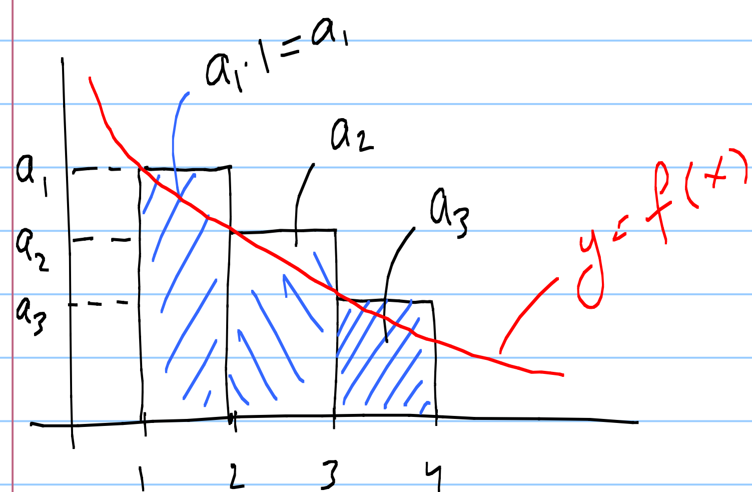
Suppose the series' terms are given by a formula $a_k = f(k)$, k - integer

Assume $f(x) \geq 0$ for $x \geq 1$,

$f(x)$ is continuous

$f(x)$ is decreasing

Idea: Approximate $\sum_{k=1}^{\infty} a_k$ by $\int_1^{\infty} f(x)\,dx$



$$a_1 \geq \int_1^2 f(x)\,dx$$

$$a_2 \geq \int_2^3 f(x)\,dx$$

$$\cdots\cdots$$

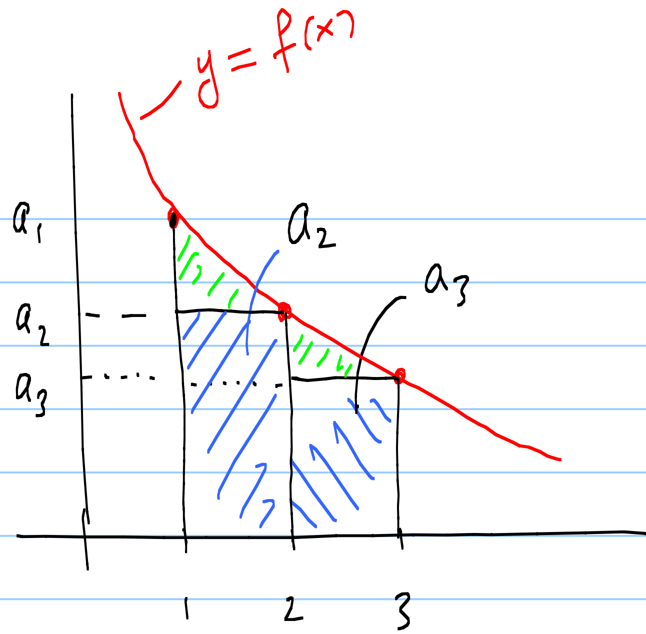
$$a_n \geq \int_n^{n+1} f(x)\,dx$$

Sum these terms:

$$S_n = \sum_{k=1}^{n} a_k \geq \int_1^{n+1} f(x)\,dx.$$

Suppose $\int_1^{\infty} f(x)\,dx$ diverges, i.e. $\int_1^{\infty} f(x)\,dx = \infty$,

then $\lim_{n\to\infty} S_n = \infty \Rightarrow \sum_{k=1}^{\infty} a_k$ diverges.



$$a_2 \le \int_1^2 f(x)\,dx$$

$$a_3 \le \int_2^3 f(x)\,dx$$

$$\cdots$$

$$a_n \le \int_{n-1}^n f(x)\,dx$$

$$S_n - a_1 = \sum_{k=2}^{n} a_k \le \int_1^n f(x)\,dx$$

$$S_n \leq a_1 + \int_1^n f(x)\,dx$$

Assume $\int_1^\infty f(x)\,dx$ converges, then

$$S_n \leq a_1 + \int_1^\infty f(x)\,dx$$

The sequence $\{S_n\}_{n=1}^\infty$ is bounded and monotonically increasing, therefore S_n converges

Then $\sum_{k=1}^\infty a_k$ converges, too.

Integral Test:

If $f(k) = a_k$, $k = 1, 2, \ldots$, $f(x) \geq 0$ if $x \geq 1$.

f is continuous and decreasing.

Then, $\int_1^\infty f(x)\,dx$ and $\sum_{k=1}^\infty a_k$ either both converge or ~~both~~ both diverge.

Example 1: $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ Convergence?

$a_k = f(k)$, where $f(x) = \frac{1}{x^2+1}$

1. $f(x) \geq 0$ if $x \geq 1$
2. $f(x)$ is continuous
3. Find $f'(x) = -\frac{1}{(x^2+1)^2} \cdot (2x) \leq 0$ if $x \geq 1$.

Therefore, f is decreasing.

We can use the integral test.

$$\int_1^\infty \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{1+x^2}\,dx =$$

$$= \lim_{R\to\infty}\left[\tan^{-1}(R) - \tan^{-1}(1)\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

since $\tan\left(\frac{\pi}{2}\right) \text{``}{=}\text{''} \infty$

$\Rightarrow$ The improper integral converges. $\Rightarrow$

The series converges.

Example $\quad p-$series, $p \neq 1$, $p > 0$

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \qquad \text{Determine convergence.}$$

Let $f(x) = \frac{1}{x^p}$, $a_k = f(k)$

1. $f(x) \geq 0$ for $x \geq 1$

2. $f(x)$ is continuous

3. $f'(x) = -p x^{-p-1} < 0$

So, we can use the integral test

$$\int_1^{\infty} x^{-p} dx = \lim_{R \to \infty} \int_1^{R} x^{-p} dx = \lim_{R \to \infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^R =$$

$$= \lim_{R\to\infty} \frac{R^{-p+1}}{-p+1} - \frac{1}{1-p} = \begin{cases} -\dfrac{1}{1-p} & \text{if } p>1 \\ \infty & \text{if } p<1. \end{cases}$$