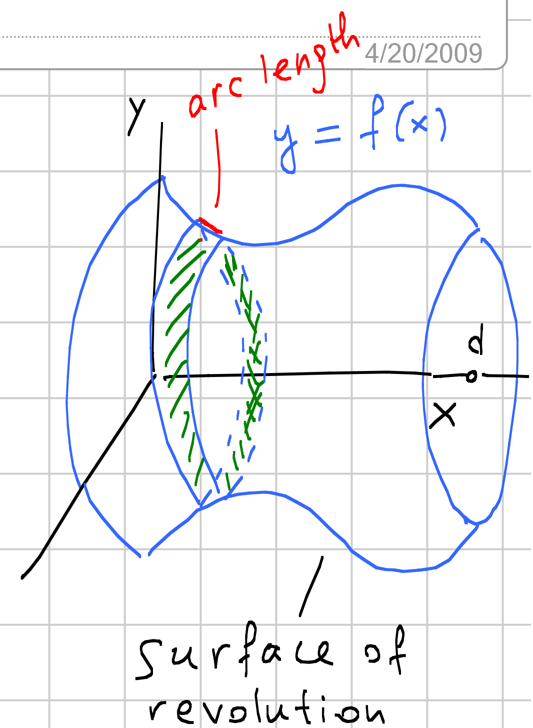
Surface Area

Note Title 4/20/2009

Surface Area =

$$\int_0^d \underbrace{2\pi \overbrace{|f(x)|}^{\text{radius}}}_{\text{length of the circle}} \underbrace{\sqrt{1+(f'(x))^2}}_{\text{arc length}}\, dx$$



Example Surface area of cone

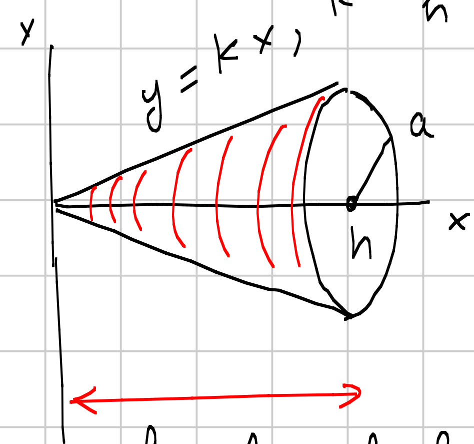
$y = kx$

$(y')^2 = k^2$

$$A = \int_0^h 2\pi |kx| \sqrt{1+k^2}\, dx = 2\pi k \sqrt{1+k^2} \int_0^h x\, dx = 2\pi k \sqrt{1+k^2}\, \frac{h^2}{2} =$$

area of cone

$$= \cancel{2}\pi \frac{a}{h} \sqrt{1+\frac{a^2}{h^2}}\, \frac{h^2}{\cancel{2}} = \boxed{\pi a \sqrt{h^2+a^2}}$$



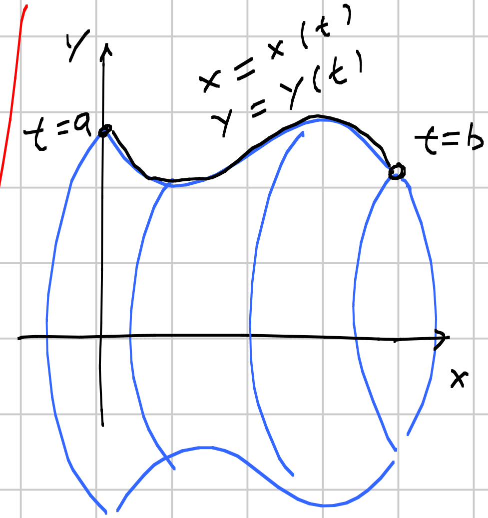
h - heigh of the cone

a - radius of base

If $h=0$, then $A = \pi a^2$ (area of the base).

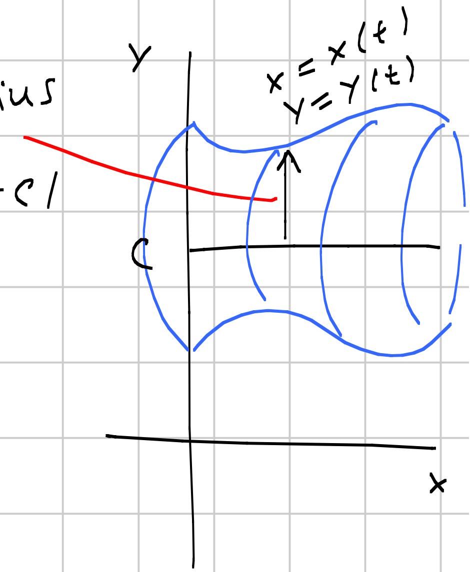
Surface area for parametric equations

$$\text{Area} = A = \int_a^b 2\pi \underbrace{|y(t)|}_{\text{radius}} \underbrace{\sqrt{x'(t)^2 + y'(t)^2}}_{\text{arc length}}\, dt$$



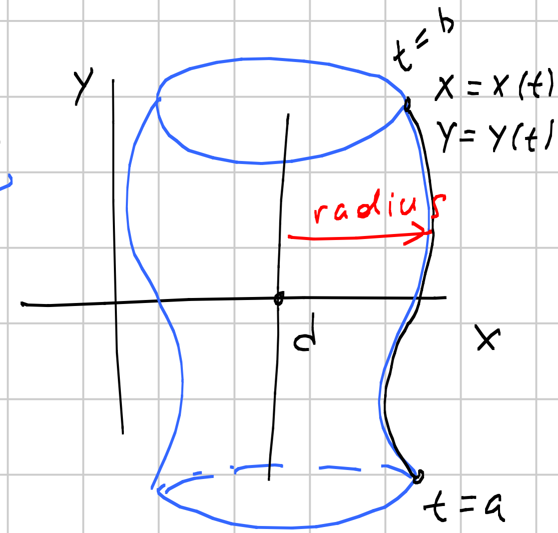
$$A = \int_a^b 2\pi \underbrace{|y(t)-c|}_{\text{radius}} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

radius
$=$
$|y(t)-c|$



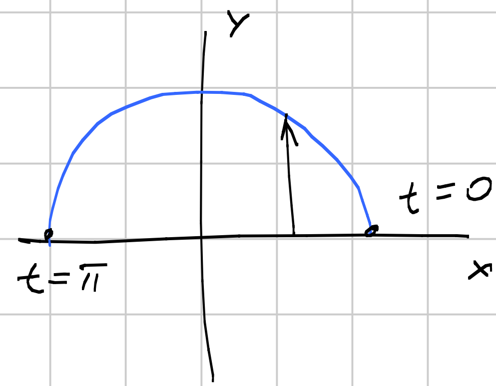
The case of axis of revolution, parallel to y-axis.

$$A = \int 2\pi \underbrace{|x(t)-d|}_{\text{radius}} \underbrace{\sqrt{x'(t)^2+y'(t)^2}\,dt}_{\text{arc length}}$$



Example | Area of the sphere

$$x = R\cos t$$
$$y = R\sin t$$



$$(x')^2 = (-R\sin t)^2 = R^2\sin^2 t$$
$$(y')^2 = (R\cos t)^2 = R^2\cos^2 t$$

$$A = \int_0^{\pi} 2\pi\,|y(t)|\,\sqrt{R^2\sin^2 t + R^2\cos^2 t}\; dt =$$

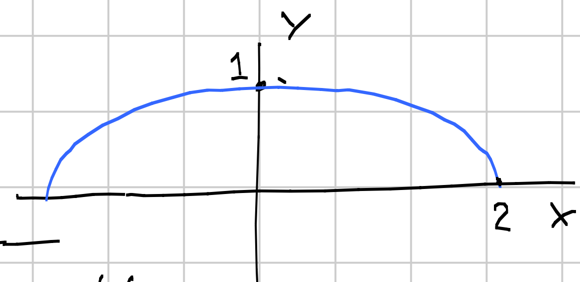
$$= \int_0^{\pi} 2\pi R\sin t\,\sqrt{R^2}\,dt = 2\pi R^2 \int_0^{\pi} \sin t\, dt =$$

$$= 2\pi R^2 (-\cos t)\Big|_0^{\pi} = 2\pi R^2 (1-(-1)) = 2\pi R^2 2 =$$

$$A \quad = 4\pi R^2$$

<u>Example</u> Ellipsoid (with 2 equal axis)

$$x = 2\cos t, \quad 0 \le t \le \pi$$
$$y = \sin t$$



$$A = \int_0^{\pi} 2\pi |\sin t| \sqrt{4\sin^2 t + \cos^2 t}\, dt =$$

$$= 2\pi \int_0^{\pi} \sin t \sqrt{1+3\sin^2 t}\, dt = \left(\begin{array}{l} u = \cos t \\ du = -\sin t\, dt \end{array} \right)$$

$\sin^2 t = 1 - \cos^2 t =$
$= 1 - u^2$

$$= 2\pi \int_1^{-1} \sqrt{1+3(1-u^2)}\, (-du) =$$

$$= 2\pi \int_{-1}^{1} \sqrt{4-3u^2}\, du$$

This can be evaluated using table of integrals.