Taylor Series

Note Title 3/16/2009

Suppose $\sum_{k=0}^{\infty} b_k (x-c)^k$ has radius of convergence $r > 0$, i.e. the series converges absolutely to some $f(x)$ for x on $(c-r, c+r)$.

$$f(x) = \sum_{k=0}^{\infty} b_k (x-c)^k = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \dots$$

Differentiate term by term

$$f'(x) = 0 + b_1 + 2b_2(x-c) + 3b_3(x-c)^2 + 4b_4(x-c)^3 + \ldots$$

$$f''(x) = 0 + 2b_2 + 2\cdot 3\, b_3(x-c) + 3\cdot 4\, b_4(x-c)^2 + \ldots$$

$$f'''(x) = 0 + 2\cdot 3\, b_3 + 2\cdot 3\cdot 4\, b_4(x-c) + \ldots$$

$$f''''(x) = 0 + 2\cdot 3\cdot 4\, b_4 + \ldots$$

Substitute $x = c$ in these relations:

$$f(c) = b_0,\quad f'(c) = b_1,\quad f''(c) = 2b_2,\quad f'''(c) = 3!\, b_3$$

$$f''''(c) = 4!\, b_4, \quad \dots, \quad f^{(k)}(c) = k!\, b_k \qquad (*)$$

If the power series converges absolutely, it converges to a function $f(x)$ with derivatives given by the above formulas.

Suppose we are given a function without the series. Can we find a series convergent to $f(x)$ using $(*)$, i.e. by taking

$$b_k = \frac{f^{(k)}(c)}{k!} .$$

Example 1 $f(x) = e^x$. Find Taylor series of e^x centered at $c = 0$.

In other words, we expand e^x about $x = 0$.

$$f'(x) = e^x, \ldots, f^{(k)}(x) = e^x$$

$$f'(0) = 1, \ldots, f^{(k)}(0) = 1$$

Taylor series

$$e^x = \sum_{k=0}^{\infty} b_k (x-c)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} x^k = \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots}_{\text{Taylor series for } e^x \text{ about } x=0.}$$

Convergence? Ratio Test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \frac{n!}{|x|^n} =$$

$$= |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1 \Rightarrow$$ the series is absolutely convergent for any x, so $r = \infty$.

Partial sum of Taylor series

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$ is Taylor polynomial of degree n.

$\boxed{\text{Convergence of Taylor series}}$

1 Suppose $f(x)$ has $n+1$ derivatives on $(c-r, c+r)$ $r>0$.

Then, for any x in $(c-r, c+r)$

$$f(x) = P_n(x) + R_n(x),$$ where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}, \text{ for some}$$

remainder

z between x and c.

2 Suppose $f(x)$ has all derivatives on $(c-r, c+r)$ and $\lim_{n\to\infty} R_n(x) = 0$, then Taylor series of $f(x)$, expanded at $x=c$, converges to $f(x)$, for all x in $(c-r, c+r)$.

Example 2 $\quad f(x) = e^x$,

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} = \frac{e^z}{(n+1)!} x^{n+1},$$

where z is between "0" and x.

Note, if $x>0$, then $0<z<x$, $e^z < e^x$

if $x<0$, then $e^z < 1$.

In both cases $e^z < \max\{1, e^x\} = M$

$$|R_n(x)| = \left| \frac{e^z}{(n+1)!} x^{n+1} \right| \le \frac{M}{(n+1)!} |x|^{n+1}$$

$$\lim_{n \to \infty} |R_n(x)| \le M \lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

for any x.