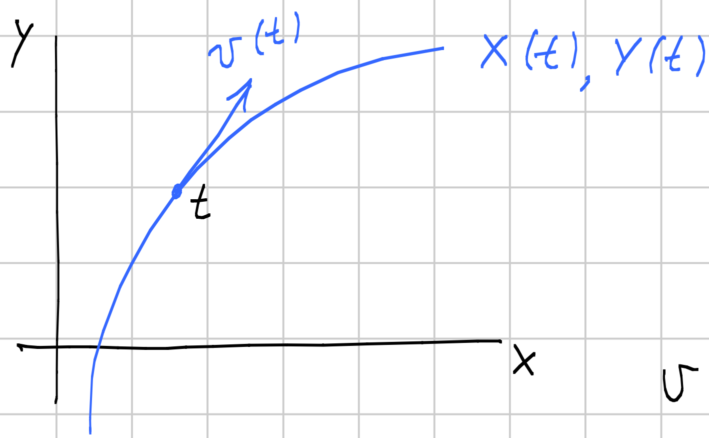
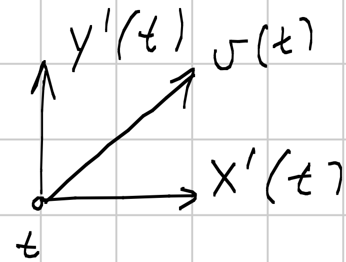
Velocity and Area by parametric eqs.

Note Title 4/15/2009

<u>Quiz tomorrow</u>: Parametric Curves Multiple choice.



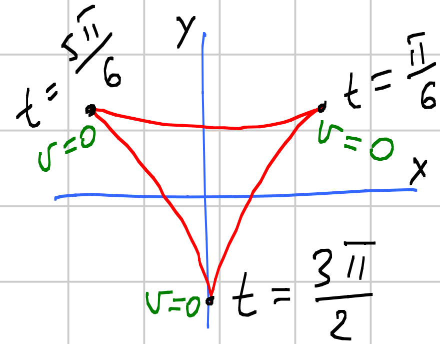


$$v(t) = \sqrt{x'(t)^2 + y'(t)^2}$$

Example (velocity of the scrambler)

$$\frac{dx}{dt} = -2\sin t + 2\cos(2t)$$

$$\frac{dy}{dt} = 2\cos t - 2\sin(2t)$$



$$v^2(t) = x'^2 + y'^2 = 4 + 4 - 8\,(\sin t\cos(2t) +$$

$$+ \cos t \sin(2t)) =$$

$$= 8 - 8\sin(3t)$$

$$v(t) = \sqrt{8 - 8\sin(3t)}$$

$\sin\alpha \cdot \cos\beta +$
$\sin\beta \cos\alpha =$
$= \sin(\alpha+\beta)$

When $v(t) = 0$?

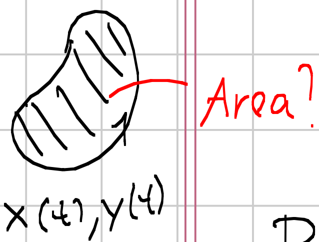
$$8 - 8\sin(3t) = 0 \Rightarrow \sin(3t) = 1$$

$$3t = \frac{\pi}{2} + 2\pi n, \quad n - \text{any integer}$$

$$3t = \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 4\pi, \frac{\pi}{2} + 6\pi$$

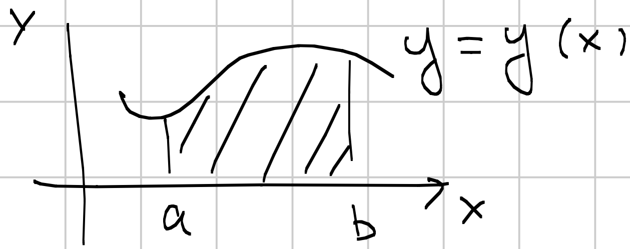
$$t = \underbrace{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}}, \quad \frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$$

At these points, velocity $v = 0$.



Area enclosed by curves

Recall



$$\text{Area} = \int_a^b y(x)\,dx,$$

Assume $y = y(x)$ is given by

$x = x(t)$, $x(c) = a$

$y = y(t)$, $x(d) = b$

change $x \to t$

$dx = x'(t)\,dt$

$$\int_a^b y(x)\,dx = \int_c^d y(x(t))\,dx(t) = $$

$$= \int_c^d y(t)\,x'(t)\,dt$$

For a closed curve without self intersections, traced out clockwise exactly once.

$$\text{Area} = \int_c^d y(t)\,x'(t)\,dt$$

$x(c) = x(d)$
$y(c) = y(d)$

$$\text{Area} = \int_c^d y(t)\,x'(t)\,dt =$$

(integrate by parts) $\quad u = y, \ v = x$

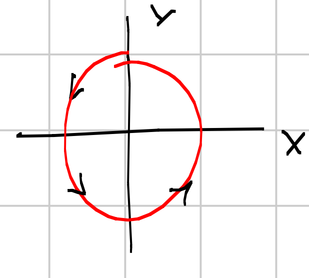
$$= \underbrace{y(t)\,x(t)\Big|_c^d}_{=\,0} - \int_c^d x(t)\,y'(t)\,dt$$

<u>Rk</u> For counterclockwise orientation, reverse the sign in both formulas.

Example 1: Circle with $R = 1$

$x = \cos t$

$y = \sin t$



$$\text{Area} = -\int_0^{2\pi} y(t)\, x'(t)\, dt =$$

$$= -\int_0^{2\pi} \sin t\,(-\sin t)\, dt = \int_0^{2\pi} \sin^2(t)\, dt =$$

$$= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\sin 2t\right) dt =$$

$$= \frac{1}{2}\int_0^{2\pi} dt + \underbrace{\cos 2t \Big|_0^{2\pi}}_{=0} = \frac{1}{2} \cdot 2\pi = \pi.$$

Arclength of parametrically given - curves.

$x(t), y(t)$

$$v(t) = \sqrt{x'(t)^2 + y'(t)^2}$$

$$S = \int_a^b v(t)\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

Example Circle

$x(t) = R\cos t$

$y(t) = R\sin t$

$$S = \int_0^{2\pi} \sqrt{R^2\sin^2(t) + R^2\cos^2(t)}\,dt =$$

$$= R \int_0^{2\pi} \sqrt{1}\, dt = 2\pi R$$

Length of scrambler curve

$$S = \int_0^{2\pi} \sqrt{8 - 8\sin 3t}\, dt .$$