Criteria of convergence of sequences

Note Title 2/18/2009

L'Hopital's Rule for sequences

Theorem 1.2 Suppose $\lim_{x \to \infty} f(x) = L$,

then $\lim_{n \to \infty} f(n) = L$

Rk 1 Converse is not true!

Example $\displaystyle\lim_{n\to\infty} \frac{n^2}{e^{2n}} \quad \left(\frac{\infty}{\infty}\right)$

$$\lim_{x\to\infty} \frac{x^2}{e^{2x}} = \lim_{x\to\infty} \frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(e^{2x})} = \lim_{x\to\infty} \frac{2x}{2e^{2x}} =$$

$$\left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{\frac{d}{dx}(2x)}{\frac{d}{dx}(2e^{2x})} = \lim_{x\to\infty} \frac{2}{4e^{2x}} = 0$$

By Thm 1.2, $\displaystyle\lim_{n\to\infty} \frac{n^2}{e^{2n}} = 0$

Theorem 1.3 **Squeeze Thm**

Let $\{a_n\}_{n=n_0}^{\infty}$, $\{b_n\}_{n=n_0}^{\infty}$, $\{c_n\}_{n=n_0}^{\infty}$

be sequences such that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L \quad \text{and}$$

for some $n_1 \geq n_0$, $\quad a_n \leq c_n \leq b_n$

Then $\lim_{n\to\infty} c_n = L$.

Example: $\left\{ \frac{\cos n}{n^3} \right\}_{n=1}^{\infty}$

Observe $-1 \leq \cos n \leq 1$, for any n.

$$\underbrace{-\frac{1}{n^3}}_{=a_n} \leq \underbrace{\frac{\cos n}{n^3}}_{=c_n} \leq \underbrace{\frac{1}{n^3}}_{=b_n}$$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, therefore by Squeeze theorem $\lim_{n \to \infty} \frac{\cos n}{n^3} = 0$.

<u>Corollary 1.1</u> If $\lim_{n \to \infty} |a_n| = 0$,

then $\lim_{n \to \infty} a_n = 0$.

Note, $-|a_n| \leq a_n \leq |a_n|$ and use the Squeeze Thm.

Recall: $n! = 1\cdot 2\cdot 3\cdots(n-1)n$

Example $\left\{\dfrac{n!}{n^n}\right\}_{n=1}^{\infty}$ Find $\lim\limits_{n\to\infty}\dfrac{n!}{n^n}$.

$$\underset{a_n}{\overset{\shortparallel}{0}} < \frac{n!}{n^n} = \underbrace{\frac{1\cdot \overset{\le 1}{\left(\frac{2}{n}\right)}\cdot \overset{\le 1}{\left(\frac{3}{n}\right)}\cdots \overset{\le 1}{\left(\frac{(n-1)}{n}\right)}\cdot \overset{\le 1}{\left(\frac{n}{n}\right)}}{n}}_{n \text{ factors}} \le \underset{b_n}{\overset{\shortparallel}{\frac{1}{n}}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0.$$

Then, by Squeeze Thm, $\lim_{n \to \infty} \frac{n!}{n^n} = 0$.

Definition 1.3 $\{a_n\}_{n=1}^{\infty}$ is increasing if $a_1 \le a_2 \le \dots \le a_n \le a_{n+1} \le \dots$

It is decreasing if

$$a_1 \ge a_2 \ge a_3 \ge \dots \ge a_n \ge \dots$$

More generally, increasing or decreasing sequences are called monotonic.

Definition 1.4 $\{a_n\}_{n=1}^{\infty}$ is bounded if
$|a_n| \le M$ for all n.

Theorem 1.4 Every bounded monotone sequence has a limit, i.e. it converges.

Example:

$$a_n = \underbrace{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot^{\sqrt{2}}}}}}}}_{n \text{ times}}$$

$a_1 = \sqrt{2} > 1$

$a_2 = \sqrt{2}^{\sqrt{2}}$

$a_3 = \sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$

a_n is increasing (monotonic)

Show that a_n is bounded.

$a_1 = \sqrt{2} < 2$

$a_2 = \sqrt{2}^{\sqrt{2}} \leq \sqrt{2}^{2} = 2$, $a_3 = \sqrt{2}^{\sqrt{2}^{\sqrt{2}}} \leq \sqrt{2}^{\sqrt{2}^{2}} = 2$

$$a_n = \sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot^{\sqrt{2}}}}}} \le \sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot^{2}}}}} = 2$$

Therefore, $a_n \le 2$, i.e. a_n is bounded.

By Theorem 1.4, a_n converges.