Integrals of rational functions

Note Title 2/2/2009

$\dfrac{P(x)}{Q(x)}$ is rational function if $P(x)$, $Q(x)$ are polynomials.

$\displaystyle\int \frac{P(x)}{Q(x)}\,dx$ can be evaluated

Using Partial Fractions

Goal: decompose $\dfrac{P(x)}{Q(x)}$ into the sum of simpler fractions:

$$\frac{c}{ax+b},\ \frac{c}{(ax+b)^2},\ \ldots,\ \frac{Ax+B}{ax^2+bx+c}$$

(1) distinct linear factors

$$\frac{P(x)}{(a_1x+b_1)(a_2x+b_2)\dots(a_nx+b_n)} =$$

$$= \frac{c_1}{a_1x+b_1} + \frac{c_2}{a_2x+b_2} + \dots + \frac{c_n}{a_nx+b_n} \quad \text{if}$$

$$\text{degree}(P(x)) < n$$

Example 1: $\int \frac{x^2-1}{x^3-9x}\,dx$

$$\frac{x^2-1}{x^3-9x} = \frac{x^2-1}{x(x-3)(x+3)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+3}$$

To find A, B, C, multiply with

x^3-9x

$$x^2-1 = A(x-3)(x+3) + Bx(x+3) + Cx(x-3)$$

$$x^2-1 = A(x^2-9) + B(x^2+3x) + C(x^2-3x)$$

$$x^2-1 = (A+B+C)x^2 + (3B-3C)x - 9A$$

$$A+B+C = 1$$

$$3B-3C = 0 \qquad B = C$$

$$-9A = -1 \rightarrow A = \frac{1}{9}$$

$$\frac{1}{9} + 2B = 1 \rightarrow B = \frac{4}{9}, \quad C = \frac{4}{9}$$

$$\int \frac{x^2-1}{x^3-9x}\,dx = \int \left(\frac{1/9}{x} + \frac{4/9}{x-3} + \frac{4/9}{x+3} \right) dx$$

$$= \frac{1}{9}\ln|x| + \frac{4}{9}\ln|x-3| + \frac{4}{9}\ln|x+3| + C$$

If degree$(P(x)) \geq n$, then first carry out long division and then use previous method.

Example 2: $\displaystyle\int \frac{x^2+x+1}{x^2-1}\,dx$

$$\begin{array}{r|l} & 1 \\ \hline x^2-1 & x^2+x+1 \\ & \underline{x^2-1} \\ & x+2 \end{array}$$

$$\frac{x^2+x+1}{x^2-1} = 1 + \frac{x+2}{x^2-1}$$

$$\frac{x+2}{x^2-1} = \frac{x+2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$x+2 = A(x+1) + B(x-1)$$

$$x+2 = (A+B)x + A - B \qquad \begin{aligned} A+B &= 1 \\ A-B &= 2 \end{aligned}$$

$$A = \frac{3}{2}, \quad B = -\frac{1}{2}$$

$$\int \frac{x^2+x+1}{x^2-1}\,dx = \int \left(1 + \frac{3/2}{x-1} - \frac{1/2}{x+1}\right) dx =$$

$$= x + \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

(2) Partial fractions: repeated linear factors

degree $(P(x)) < n$

- $$\frac{P(x)}{(ax+b)^n} = \frac{c_1}{ax+b} + \frac{c_2}{(ax+b)^2} + \dots + \frac{c_n}{(ax+b)^n}$$

- $$\frac{P(x)}{(a_1x+b_1)^{n_1}(a_2x+b_2)^{n_2}} = \frac{c_1}{(a_1x+b_1)} + \dots + \frac{c_{n_1}}{(a_1x+b_1)^{n_1}} + \frac{d_1}{(a_2x+b_2)} + \dots + \frac{d_{n_2}}{(a_2x+b_2)^{n_2}}$$

Example 3: $\int \frac{1}{(x-1)(x+1)^2}\, dx$

$$\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Multiply with $(x-1)(x+1)^2$:

$$1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$A = \frac{1}{4}, \quad B = -\frac{1}{4}, \quad C = -\frac{1}{2}$$

$$\int \frac{1}{(x-1)(x+1)^2}\,dx = \int \left(\frac{1/4}{x-1} - \frac{1/4}{x+1} - \frac{1/2}{(x+1)^2} \right) dx$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2}\,\frac{1}{x+1} + C.$$