Sequences of real numbers

Note Title 2/16/2009

Definition 1: Sequence is a function defined on positive integers:

$$f : \{1, 2, 3, \dots, n, \dots\} \to \{f(1), f(2), \dots, f(n), \dots\}$$

Example 1: $f(n) = \frac{1}{n}$, $n = 1, 2, 3, \dots,$

Sequence can be defined by a formula.

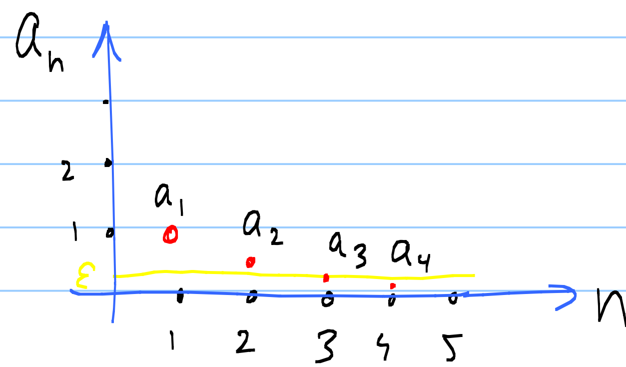
Example 2: Sequence of temperature recordings, defined by a table

day	1	2	3
temperature	70°	75°	65°

Notation: a_n = general n-th term

e.g. in Example 1, $a_n = \frac{1}{n}$

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\}$$



– graph of a sequence

<u>Note:</u> $a_n = \frac{1}{n}$ approaches "0" as $n \to \infty$.

<u>Definition 2</u>: A sequence $\{a_n\}_{n=1}^{\infty}$ converges to L if for any $\varepsilon > 0$, there is an integer $N > 0$ such that

$$|a_n - L| < \varepsilon \quad \text{whenever} \quad n > N.$$

Otherwise, the sequence diverges.

Use Def 2 to prove that $a_n = \frac{1}{n}$ converges to "0".

Take $L=0$, let $\varepsilon>0$. Take N to be an integer such that $N > \frac{1}{\varepsilon}$.

Then, $|a_n - L| = \left|\frac{1}{n} - 0\right| = \frac{1}{n}$

If $n>N$, then $\frac{1}{n} < \frac{1}{N} < \varepsilon$.

But then, $|a_n - L| < \varepsilon$.

Notation: If L is the limit of

$$\{a_n\}_{n=1}^{\infty}, \qquad \lim_{n \to \infty} a_n = L$$

Theorem (Properties of lim)

1. $\lim_{n \to \infty} (a_n \pm b_n) = \lim_{n \to \infty} a_n \pm \lim_{n \to \infty} b_n$

2. $\lim_{n \to \infty} (a_n \cdot b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n$

3. $$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} \quad \left(\lim_{n\to\infty} b_n \neq 0\right)$$

Example 3:

$$a_n = \frac{n^2+n-1}{3n^2-1}$$

$$\lim_{n\to\infty}\frac{n^2+n-1}{3n^2-1} = \lim_{n\to\infty}\frac{1+\frac{1}{n}-\frac{1}{n^2}}{3-\frac{1}{n^2}} =$$

Divide both parts of the fraction with highest power of n, i.e. n^2

$$= \frac{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}-\frac{1}{n^2}\right)}{\lim\limits_{n\to\infty}\left(3-\frac{1}{n^2}\right)} = \frac{\lim 1 + \lim \frac{1}{n} - \lim \frac{1}{n^2}}{\lim 3 - \lim \frac{1}{n^2}} =$$

$$= \frac{1+0-0}{3-0} = \frac{1}{3}$$ ← denominator $\neq 0$, so we could use (3).

Example 4: (Divergent sequences)

$a_n = n$, show that this sequence diverges.

Indeed, assume L is a limit. Let $\varepsilon = 1 > 0$, then no matter what N is, there will be terms a_n, $n > N$, such that $|a_n - L| > \varepsilon$.

(2) $a_n = (-1)^n$, $\{-1, 1, -1, 1, \dots\}$.

Show that this sequence diverges.