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Invariant Tori in Hamiltonian Systems with Impacts

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Abstract: It is shown that a large class of solutions in two-degree-of-freedom Hamiltonian systems of billiard type can be described by slowly varying one-degree-of-freedom Hamiltonian systems. Under some non-degeneracy conditions such systems are found to possess a large set of quasiperiodic solutions filling out two dimensional tori, which correspond to caustics in the classical billiard. This provides a unified proof of existence of quasiperiodic solutions in convex billiards and other systems with impacts including classical billiard in electric and magnetic fields, dual billiard, and Fermi–Ulam systems.

1. Introduction

1.1. Billiards and systems with impacts. The classical billiard system describes the free motion of a particle in a planar region bounded by a closed curve. The particle moves along a straight line and is reflected from the boundary according to the rule "the angle of reflection equals the angle of incidence". The systematic study of classical billiards was started by Birkhoff to illustrate and develop certain concepts in the theory of Hamiltonian Dynamical systems with two degrees of freedom [5]. Since then the billiard has become a basic model in such diverse fields as Foundations of Statistical Mechanics, Ergodic Theory, Quantum Chaos, etc. The billiards represent the simplest systems in Classical Mechanics which still exhibit any kind of behavior observed in two-degree-of-freedom Hamiltonian systems.

One of the most important results concerning billiards was the discovery of caustics clustering at the boundary of a smooth convex billiard with non-vanishing curvature by Lazutkin [16,17]. In geometric optics a caustic is defined as the envelope of a light ray trajectory, so that any ray tangent to a caustic remains tangent to it after reflection from the boundary. The presence of a caustic implies non-ergodic behavior as the corresponding invariant curve (family of rays tangent to a caustic) separates the phase space into invariant components. On the other hand, caustics can be used to estimate eigenvalues and to construct quasimodes in the corresponding Dirichlet problem, see [16].

In his original proof Lazutkin showed the existence of caustics by reducing the billiard ball map[1] to a near integrable form, and by applying KAM theory to obtain invariant curves.

The near integrable behavior in the vicinity of the boundary of a smooth convex billiard can be anticipated by observing that a trajectory nearly tangent to the boundary will experience many collisions with it before the curvature significantly changes. Such observation raises a hope of introducing different "time scales" and obtaining an adiabatic invariant. However, in the proofs by Lazutkin and others, see e.g. [3,6,17,18], this simple physical intuition is hidden because the billiard ball map is used.

In this paper, we prove the existence of caustics in systems of billiard type by using Arnold's result on the existence of invariant tori in smooth slowly varying oscillatory Hamiltonian systems [2], which allows us to make the above physical argument rigorous without sacrificing its clarity. In order to apply Arnold's approach to the systems of billiard type, we use the Hamiltonian formalism for the systems with unilateral constraint developed in [13] and in subsequent papers by Markeev, Ivanov and their coauthors. The main idea of their approach is that the Hamiltonin systems with impacts can be largely treated as smooth Hamiltonian systems.

Here, we start with the "billiard" Hamiltonian, which is nonsmooth at the boundary surface, and following Markeev [23] we apply the isoenergetic reduction in the presence of a unilateral constraint. After carrying out a canonical rescaling we obtain a slowly periodically varying Hamiltonian system with one degree of freedom. The obtained nonsmooth Hamiltonian function is reduced to a near integrable form following the well known procedure, see e.g. [2,20]. The vector field generated by the near integrable Hamiltonian induces a near integrable mapping on the surface of the section corresponding to the boundary. This map is smooth (for the billiard flow is smooth away from the boundary) and satisfies the conditions of Moser's small twist theorem, which implies the existence of invariant curves corresponding to caustics. In a similar situation, KAM theory has been applied to a system with unilateral constraint in [22]. To summarize, we have applied the approach in [2], using a non-smooth version of the Hamiltonian formalism from [13] to the billiard systems. This is possible because the averaging transformations can be applied to nonsmooth Hamiltonian functions.

Next, this approach is applied to a larger class of non-smooth Hamiltonian functions (Subsect. 1.3), which provides a unified stability proof for various systems of billiard type such as the Fermi–Ulam oscillator, dual billiard, and billiard in magnetic and electric fields (Sect. 2). Finally, in Sect. 3, we provide an example of instability due to Halpern [12] in the framework of the Hamiltonian approach for non-smooth systems [13] and give an improved criterion for the billiard flow to be well defined.

Whenever appropriate, we assume that Hamiltonian functions are analytic in order not to burden the exposition with the estimates. All statements can be extended to finitely differentiable functions.

1.2. Caustics in convex billiards. In this section, we use the above approach to give a new proof of Lazutkin's theorem on existence of caustics in convex billiards.

Introducing the boundary coordinates (r, s) as in [17], where r is the distance from the boundary and s is the natural parameter along the boundary curve, we obtain the

[1] The map that associates to an outgoing ray's pair of the reflection point and the angle of reflection the corresponding data after the next reflection, see e.g. [14].

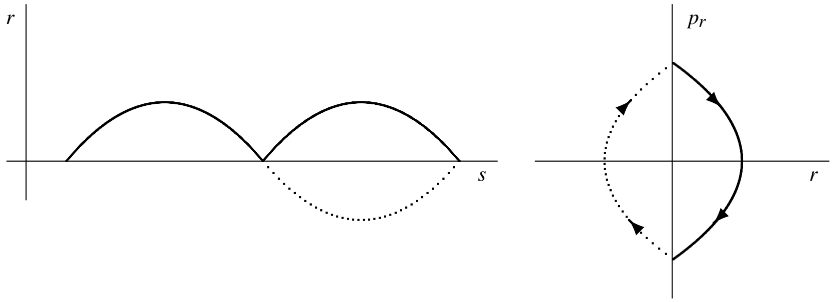

Fig. 1. The trajectory of the billiard ball in the configuration space and in the reduced phase space

Lagrangian of a free particle in the new coordinates[2]

$$L = \frac{\dot{q}_1^2}{2} + \frac{\dot{q}_2^2}{2} = \frac{\dot{r}^2}{2} + (1 - k(s)r)^2 \frac{\dot{s}^2}{2}.$$

Carrying out the Legendre transformation we obtain the Hamiltonian

$$H = \frac{p_r^2}{2} + \frac{p_s^2}{2(1 - k(s)r)^2}, \tag{1}$$

which also describes the motion away from the boundary. Following Ivanov and Markeev [13], we modify the Hamiltonian so that it would describe full dynamics, including the impacts with the boundary, by letting r assume negative values and substituting $|r|$ instead of r,

$$H = \frac{p_r^2}{2} + \frac{p_s^2}{2(1 - k(s)|r|)^2}. \tag{2}$$

The equivalence of these systems is easy to see from Fig. 1, see also [6], where such a description is used for a billiard problem.

Since the system is autonomous, the energy does not change and, therefore, it is reasonable to carry out isoenergetic reduction. Using an invariant relation of the Hamiltonian vector field with the 1-form $p_r dr + p_s ds - H dt = p_r dr + H d(-t) - (-p_s) ds$, we choose $K = -p_s$ as the new Hamiltonian and s as the new time

$$K = -(1 - k(s)|r|)\sqrt{2H - p_r^2}, \tag{3}$$

where the sign of the square root is chosen to be positive, which corresponds to the motion in the positive direction of s. Thus, we have obtained a one-degree-of-freedom nonautonomous system. Taking $H = 1/2$ and rescaling the system[3]

$$K = \varepsilon^2 F \quad p_r = \varepsilon P_R,$$
$$s = \varepsilon S \qquad r = \varepsilon^2 R,$$

[2] This change of coordinates is well defined for $r < \frac{1}{k_{\max}} < \frac{1}{k}$, but since we are interested in the solutions staying near the boundary, this is not a serious restriction.

[3] The rescaling is motivated by an elementary geometric observation: if the angle of reflection is of order ε (i.e. $p_r \sim \varepsilon$) then the arc length between the two successive collisions is also of order ε (i.e. $s \sim \varepsilon$) and the light ray stays ε^2-close to the boundary (i.e. $r \sim \varepsilon^2$).

we obtain the new Hamiltonian

$$F = -(\varepsilon^{-2} - k(\varepsilon S)|R|)\sqrt{1 - \varepsilon^2 P_R^2}. \tag{4}$$

Expanding the square root in Taylor series, we obtain

$$F = \frac{P_R^2}{2} + k(\varepsilon S)|R| + \varepsilon^2 F_2(|R|, P_R, \varepsilon S, \varepsilon), \tag{5}$$

where $F_1(|R|, P_R, \varepsilon S, \varepsilon)$ is a real analytic function for $(R \neq 0, |P_R| < \varepsilon^{-1})$. Using more convenient notation $R = x$, $P_R = y$, $S = t$, $F = H$, we rewrite the Hamiltonian

$$H = \frac{y^2}{2} + k(\lambda)|x| + \varepsilon^2 H_2(|x|, y, \lambda, \varepsilon), \tag{6}$$

where $\lambda = \varepsilon t$. The leading term of the Hamiltonian represents a slowly varying one-degree-of-freedom Hamiltonian system, which corresponds to the system of a bouncing ball in a slowly varying gravity field. If the Hamiltonian function were smooth we would be able to apply Arnold's theorem on perpetual conservation of adiabatic invariant [2]. But since the Hamiltonian is not smooth in x we will follow the reduction procedure in [2], pointing out how it modifies for the non-smooth case according to [22].

We first apply the action-angle transformation as in [22] $(x, y, \lambda) \to (\phi, I, \lambda)$ with the Hamiltonian $H_0 = \frac{y^2}{2} + k(\lambda)|x|$. The details of the derivation can be found in [1] or in the next section.

The action variable I is given by the area enclosed with the corresponding trajectory of the autonomous system $H_0(|x|, y, \lambda)$ with the frozen parameter λ,

$$I(x, y, \lambda) = I(H(x, y, \lambda), \lambda) = 4\sqrt{2}\int_0^{\frac{H}{k}} \sqrt{H - kx}\,dx = \frac{2\sqrt{2}}{3k}H^{3/2}. \tag{7}$$

The angular variable $\phi(x, y, \lambda)$ is given by the time it takes the solution in the autonomous system to move from the section $((x = 0, y > 0))$ to (x, y), divided by the period of one revolution $T(H, \lambda)$ in the autonomous system.

The new Hamiltonian takes the form

$$H = \left[\frac{3}{2\sqrt{2}}k(\varepsilon t)I\right]^{\frac{2}{3}} + \varepsilon H_1(I, |\phi|, \varepsilon t, \varepsilon), \tag{8}$$

where H_1 is real analytic (see the next section). Because of the reflectional symmetry in the system and our choice of $\phi = 0$ at $x = 0$, $y > 0$ the Hamiltonian depends on $|\phi|$.

Even though the obtained Hamiltonian is a small perturbation of an integrable one, KAM theory still cannot be applied due to the explicit time dependence in the leading term. Now, we make the leading term of the Hamiltonian time independent.

Since the integral curves of the Hamiltonian system are invariantly associated with the differential form

$$I d\phi - H(I, \phi, \lambda)dt = -\frac{1}{\varepsilon}\{H d\lambda - \varepsilon I(H, \lambda, \phi, \varepsilon)d\phi\},$$

where

$$I(H, \lambda, \phi, \varepsilon) = \frac{2\sqrt{2}}{3}\frac{H^{3/2}}{k(\lambda)} + \varepsilon I_1(H, \lambda, \phi, \varepsilon) \tag{9}$$

is the inverse function of (8), we can choose $(\varepsilon I(H, \lambda, \phi, \varepsilon), \phi, H, \lambda)$ as a new Hamiltonian, time, momentum, and position, respectively.

Now, we introduce a linear "time-dependent" transformation which will make the leading term of the Hamiltonian $\varepsilon I(H, \lambda, \phi, \varepsilon)$ $\lambda-$ independent[4],

$$h = \frac{\langle k^{2/3} \rangle}{k^{2/3}(\lambda)} H \quad \tau = \frac{1}{\langle k^{2/3} \rangle} \int_0^{\lambda} k^{2/3}(\lambda) d\lambda.$$

The new Hamiltonian takes the form

$$J(h, \tau, \phi, \varepsilon) = \varepsilon J_0(h) + \varepsilon^2 J_1(h, \tau, |\phi|, \varepsilon),$$

where

$$J_0(h) = \frac{2\sqrt{2}}{3} \frac{h^{3/2}}{\langle k^{2/3} \rangle^{3/2}}$$

and J_1 is 1-periodic in τ. The corresponding equations of motion take the form

$$\begin{cases} \dfrac{d\tau}{d\phi} = \varepsilon J_0'(h) + \varepsilon^2 \dfrac{\partial J_1}{\partial h}(h, \tau, |\phi|, \varepsilon) \\ \dfrac{dh}{d\phi} = -\varepsilon^2 \dfrac{\partial J_1}{\partial \tau}(h, \tau, |\phi|, \varepsilon). \end{cases}$$

Finally, we let $\nu = J_0'(h)$ and integrate the equations of motion on $\phi \in (0, 1/2)$ obtaining a monotone twist map

$$\begin{cases} \tau_1 = \tau_0 + \varepsilon \nu_0 + \varepsilon^2 Q_1(\tau_0, \nu_0, \varepsilon) \\ \nu_0 = \nu_1 + \varepsilon^2 Q_2(\tau_0, \nu_0, \varepsilon), \end{cases} \tag{10}$$

which satisfies the conditions of Moser's small twist theorem [24]. Applying the theorem in the annulus $1 \le \nu \le 2$ we obtain a large set of invariant circles and the measure of the complement of their union tends to zero as $\varepsilon \to 0$. Retracing the transformations we find that the subset of the billiard table

$$U^{\varepsilon} = \left\{ (r, s) \in R^+ \times S^1 \mid \varepsilon^2 C_1 k^{-1/3}(s) \le r \le \varepsilon^2 C_2 k^{-1/3}(s) \right\},$$

where $0 < C_1 < C_2$, is filled with caustics and the relative measure of the complement of their union tends to zero as $\varepsilon \to 0$.

Now, consider a sequence of subsets of the billiard table $U^n = U^{\varepsilon_n}$, $n \in N$, where $\varepsilon_n = (\sqrt{C_1/C_2})^n \varepsilon_0$. It is easy to see that $\bigcup_{n=1} U^n$ is a neighborhood of the boundary and $\varepsilon_n \to 0$. Therefore, the caustics accumulate at the boundary and the relative measure of the complement of their union goes to zero near the boundary.

4 It is obtained by taking $H = ck^{2/3}h$, so that the leading term in (9) would be λ-independent, using invariance of the 2-form $dH \wedge d\lambda = dh \wedge d\tau$, and applying a normalized periodicity condition: $\tau = 0$ if $\lambda = 0$ and $\tau = 1$ if $\lambda = 1$.

1.3. Invariant tori in the systems of billiard type. In this section we state and prove the theorem on existence of invariant tori in the slowly periodically varying Hamiltonian systems generalizing (6) and to which many systems of billiard type can be reduced. The proof follows the argument in the preceding section.

We assume that a slowly periodically varying oscillatory conservative system is described by the Hamiltonian function $H = H_0(|x|, y, \lambda) + \varepsilon H_1(|x|, y, \lambda, \varepsilon)$ $(\lambda = \varepsilon t)$. Introducing the action-angle variables for $H = H_0(|x|, y, \lambda)$ as before, we let $I(x, y, \lambda)$ be the area enclosed by the curve $H_0(|\xi|, \eta, \lambda) = H_0(|x|, y, \lambda)$ and $\phi(x, y, \lambda)$ be the time it takes the solution with $H = H_0(x, y, \lambda)$ to travel from $\xi = 0$ to $\xi = x$ in the autonomous system $H_0(|\xi|, \eta, \lambda)$ divided by the period $T(H, \lambda)$. More formally,

$$I(x, y, \lambda) = I(H(x, y, \lambda), \lambda) = \oint_{H=H_0(|x|,y,\lambda)} y\,dx, \tag{11}$$

and in a neighborhood of (x_0, y_0, λ_0) the transformation can be obtained from the generating function

$$S(x, I, \lambda) = \int_{C(x,x_0)} y\,dx, \tag{12}$$

where C is a part of the level curve $H_0(|x|, y, \lambda) = H_0(I, \lambda)$ and $H_{0y}(|x_0|, y_0, \lambda_0) \neq 0$. If, however, $H_{0y}(|x_0|, y_0, \lambda_0) = 0$, then the generating function $S(y, I, \lambda)$ has to be used.

The generating function and the action-angle transformation is obtained using the relation between the 1-form

$$ydx - Hdt = Id\phi - Kdt + dS = -\phi dI - Kdt + dS(x, I, t)$$

and the vector field [1]. Indeed, for fixed I and frozen λ we obtain (12). Using the above differential relation we also obtain $\phi = \frac{\partial S}{\partial I}(x, I, t)$ and since $\phi \in [0, 1]$ is an angle we obtain (11) (i.e. S must increase by the value of I over each rotation).

The new Hamiltonian takes the form

$$\begin{aligned} H &= H_0(I, \lambda) + \varepsilon H_1(I, \phi, \lambda, \varepsilon) + \varepsilon S_\lambda(x(I, \phi, \lambda), I, \lambda) \\ &= H_0(I, \lambda) + \varepsilon \tilde{H}_1(I, \phi, \lambda, \varepsilon). \end{aligned} \tag{13}$$

Theorem 1.1. *Assume that the surfaces $H_0(|x|, y, \lambda) = H_0(I, \lambda)$ are homeomorphic to 2-tori on an open interval $I \in (I_1, I_2)$, fill out an open domain Ω and the following conditions hold:*

$$H(\rho, y, \lambda, \varepsilon) \quad \text{is 1-periodic in } \lambda \text{ and real analytic in all variables in } \Omega^+ \times [0, \varepsilon_0), \quad \text{where } \Omega^+ = \Omega \cap \{x > 0\},$$

$$\omega(I, \lambda) = \frac{\partial H_0}{\partial I}(I, \lambda) \neq 0,$$

$$\frac{d\bar{\omega}}{dI} = \frac{\partial^2}{\partial I^2} \oint H_0(I, \lambda)\,d\lambda \neq 0,$$

$$\frac{\partial H_0}{\partial y}(0, y, \lambda) \neq 0$$

everywhere in the toroidal layer $I \in (I_1, I_2)$. Then for sufficiently small ε the above layer possesses invariant tori and the relative measure of the complement of their union tends to zero as $\varepsilon \to 0$.

Proof. We proceed as in the proof of Lazutkin's theorem by first showing that the Hamiltonian in the action-angle variables takes the form

$$H(I,\lambda,\phi,\varepsilon) = H_0(I,\lambda) + \varepsilon \tilde{H}_1(I,\lambda,|\phi|,\varepsilon) \tag{14}$$

and is real analytic in $(I,\lambda,|\phi|,\varepsilon)$ if $\phi \neq 0, \frac{1}{2}$. Then applying the Implicit Function Theorem we obtain that the inverse function

$$I(H,\lambda,\phi,\varepsilon) = I_0(H,\lambda) + \varepsilon I_1(H,\lambda,|\phi|,\varepsilon) \tag{15}$$

is real analytic in $(H,\lambda,|\phi|,\varepsilon)$ if $\phi \neq 0, \frac{1}{2}$. Finally, carrying out an averaging transformation $(H,\lambda) \to (h,\tau)$ similar to the one in the previous section we obtain a near integrable Hamiltonian

$$J(h,\tau,\phi,\varepsilon) = \varepsilon J_0(h) + \varepsilon^2 J_1(h,\tau,|\phi|,\varepsilon), \tag{16}$$

which is real analytic in $(h,\tau,|\phi|,\varepsilon)$ if $\phi \neq 0, \frac{1}{2}$.

We start with

Proposition 1.1. *Under the conditions of the theorem $H_0(I,\lambda)$ is an analytic function.*

Proof. First, consider $I(H,\lambda)$, given by the area enclosed with $H = H_0(|x|, y, \lambda)$. Fixing $H = H_0$ and $\lambda = \lambda_0$ so that $I_1 < I(H_0,\lambda_0) < I_2$ we show that $I(H,\lambda)$ is analytic in a neighborhood of (H_0,λ_0). Indeed, consider the right part of the boundary $(x \geq 0)$ as a union of arcs (z_k, z_{k+1}), where $z_k = (x_k, y_k)$, $k = 1, 2, ..., N$, such that $H_y(z_k) \neq 0$ and $H_x(z_k) \neq 0$ and in each arc either $H_x \neq 0$ or $H_y \neq 0$. Therefore, each arc can be represented as $y(x,H,\lambda)$ or $x(y,H,\lambda)$ and we have

$$\begin{aligned}\frac{1}{2}[I(H,\lambda) - I(H_0,\lambda_0)] = &\sum_{k,k+1\in A_y} \int_{x_k}^{x_{k+1}} [y(x,H,\lambda) - y(x,H_0,\lambda_0)]\,dx \\ &+ \sum_{k,k+1\in A_x} \int_{y_{k+1}}^{y_k} [x(y,H,\lambda) - x(y,H_0,\lambda_0)]\,dy \\ &+ \sum_{k\notin A_x\cup A_y} \int_{x_k}^{x(y_k,H,\lambda)} [y(x,H,\lambda) - y_k]dx,\end{aligned}$$

where $k \in A_y (k \in A_x)$ if the arcs having z_k as a boundary point are such that $H_y \neq 0 (H_x \neq 0)$. It is easy to see that all terms in the sums are real analytic functions. Therefore $I(H,\lambda)$ is also real analytic. This implies analyticity of $H_0(I,\lambda)$ by the Implicit Function Theorem, which can be applied in Ω since $\frac{\partial H_0}{\partial I}(I,\lambda) \neq 0$. □

Now, we show that the Hamiltonian function (14) is analytic. In a neighborhood of (x_0, y_0, λ_0) the transformation $(x,y,\lambda) \to (\phi, I, \lambda)$, where $x_0 x \geq 0$ and $H_{0y}(x_0, y_0, \lambda_0) \neq 0$, is defined implicitly by

$$\begin{aligned}\phi &= \phi_0 + \frac{\partial S_{x_0}}{\partial I}(x, I, \lambda),\\ y &= \frac{\partial S_{x_0}}{\partial x}(x, I, \lambda),\end{aligned}$$

where the generating function S_{x_0} is given by

$$S_{x_0}(x, I, \lambda) = \int_{x_0}^{x} y(|\xi|, H_0(I, \lambda), \lambda)\, d\xi.$$

In case $H_{0y}(x_0, y_0, \lambda_0) = 0$ then $H_{0x}(x_0, y_0, \lambda_0) \neq 0$[5] and the transformation can be defined implicitly by

$$\phi = \phi_0 + \frac{\partial S_{y_0}}{\partial I}(y, I, \lambda),$$
$$x = -\frac{\partial S_{y_0}}{\partial y}(y, I, \lambda),$$

where the generating function S_{y_0} is given by

$$S_{y_0}(y, I, \lambda) = \int_{y_0}^{y} x(\eta, H_0(I, \lambda), \lambda)\, d\eta.$$

Thus, we can always choose a generating function which defines the transformation locally [1].

In a neighborhood of any point different from $(0, y_0, \lambda_0)$ the transformation is real analytic. Indeed, the generating function is analytic for it is an integral of an analytic function. If the transformation is generated by S_{x_0}, then we can invert the equation for ϕ to obtain $x = x(I, \phi, \lambda)$ and substitute it in the second equation for y. The obtained explicit canonical transformation is analytic and invertible for its Jacobian is equal to one. The same argument applies to the transformation generated by S_{y_0}.

Because of the symmetry of the Hamiltonian in x, we obtain that $x(I, -\phi, \lambda) = -x(I, \phi, \lambda)$, therefore $|x(I, -\phi, \lambda)| = x(I, |\phi|, \lambda)$ and $y(I, \phi, \lambda) = y(|x(I, \phi, \lambda)|, H_0(I, \lambda), \lambda) = y(|\phi|, I, \lambda)$. Thus, $H_1(|x|, y, \lambda, \varepsilon)$ will take the form $\varepsilon H_1(|\phi|, I, \lambda, \varepsilon)$. Since S is antisymmetric in x and x is antisymmetric in ϕ then $\partial_\lambda S(x, I, \lambda)$ is symmetric in ϕ. Thus, we have proven that the Hamiltonian in action-angle variables is given by (14) and is real analytic.

The averaging transformation $(H, \lambda) \to (h, \tau)$ is defined similar to the action-angle transformation: h is given by the area under the curve $I = I_0(H, \lambda)$,

$$h(H, \lambda) = h(I_0(H, \lambda)) = \int_0^1 H_0(I_0(H, \lambda), \alpha)\, d\alpha,$$

and the generating function is given by

$$W(h, \lambda) = \int_0^{\lambda} H(J_0(h), \xi)\, d\xi,$$

where $J_0(h)$ is the inverse of $h(I)$ and $\tau = \partial_h W(h, \lambda)$.

It is easy to check that the transformation preserves periodicity: if $(\lambda, H) \to (\tau, h)$ then $(\lambda + 1, H) \to (\tau + 1, h)$. The transformation is real analytic because $H(I_0, \lambda)$ is analytic and $\partial_I H(I, \lambda) \neq 0$. Therefore, the new Hamiltonian function (16) is also real analytic, if $\phi \neq 0, \frac{1}{2}$.

Proceeding as in the previous section we obtain the map (10) which satisfies the conditions of Moser's small twist theorem and thus, has a large set of invariant circles. The rest of the proof follows the argument at the end of the last section.

[5] The frequency does not vanish in Ω and therefore $\nabla H \neq 0$.

Remark 1.1. If the Hamiltonian $H(|x|, y, \lambda)$ is quasiperiodic in λ then after similar transformations one obtains a monotone twist map (10) which is quasiperiodic in τ. A similar result then can be obtained using a quasiperiodic version of the monotone twist map, see e.g. [32].

Corollary 1.1. *Under the conditions of the theorem the action variable is a perpetually conserved adiabatic invariant. More precisely for any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ and $I_1 + \delta \le I_0 \le I_2 - \delta$, then $|I(t) - I(0)| \le \delta$.*

Corollary 1.2. *If all conditions of the above theorem are satisfied except for periodicity of $H(x, y, \lambda)$ in λ then the action variable is an adiabatic invariant, i.e. the statement in the above Corollary is true for $|t| \le C\varepsilon^{-1}$.*

2. Applications

In this section we consider various systems to which the above theorem applies.

2.1. Billiard in constant magnetic and electric fields. The billiards in magnetic field were considered in [27] and later in [3,4]. We use the above theorem to provide a criterion of stability of the solutions near the boundary.

The Lagrangian of the problem is given by

$$L = \frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2} + A(x, y)\dot{x} - W(x, y).$$

Introducing the boundary coordinates we obtain the Lagrangian [6]

$$L = \frac{\dot{r}^2}{2} + (1 - k(s)r)^2 \frac{\dot{s}^2}{2} + M(r, s)\dot{s} - V(r, s), \tag{17}$$

where $M(r, s)$ and $V(r, s)$ are related to $A(x, y)$ and $W(x, y)$.

Carrying out the Legendre transformation we obtain the Hamiltonian

$$H = \frac{p_r^2}{2} + \frac{(p_s - M)^2}{2(1 - k(s)|r|)^2} + V, \tag{18}$$

where we have exchanged r for $|r|$ as before to account for collision with the boundary. Using the invariance of the form $p_r dr + p_s ds - H dt = p_r dr + H d(-t) - (-p_s) ds$ we choose $K = -p_s$ as the new Hamiltonian and s as the new time

$$K = -M - (1 - k(s)|r|)\sqrt{2H - p_r^2 - 2V}, \tag{19}$$

where the square root is taken with positive sign. Expanding M in series of r we have $M(r, s) = M_1(s)r + M_2(s)r^2 + \cdots$, and rescaling the system

$$K = \varepsilon^2 F, \quad p_r = \varepsilon P_R,$$
$$s = \varepsilon S, \qquad r = \varepsilon^2 R,$$

[6] We can neglect the other term linear in velocity $N\dot{r}$, for it can be absorbed in $M\dot{s}$ since addition of a full time-derivative to the Lagrangian does not effect the equations of motion.

we obtain the new Hamiltonian

$$F = - M_1(\varepsilon S)|R| - (\varepsilon^{-2} - k(\varepsilon S)|R|)\sqrt{2H - 2V_0(\varepsilon S) - \varepsilon^2 P_R^2 - \varepsilon^2 2V_1(\varepsilon S)|R| + O(\varepsilon^4)} + O(\varepsilon^2). \tag{20}$$

Expanding the square root and rearranging the terms we obtain

$$F = \frac{P_R^2}{2a(\varepsilon S)} + b(\varepsilon S)|R| + \varepsilon^2 F_1(|R|, P_R, \varepsilon S, \varepsilon), \tag{21}$$

where

$$a(\varepsilon S) = \sqrt{2H - 2V_0(\varepsilon S)},$$
$$b(\varepsilon S) = k\sqrt{2H - 2V_0(\varepsilon S)} - M_1(\varepsilon S) + \frac{V_1(\varepsilon S)}{\sqrt{2H - 2V_0(\varepsilon S)}}.$$

To keep the expression under the square root positive we require $H > \max V_0(s)$ therefore $a > 0$.

Introducing the standard notation $R = x$, $P_R = y$, $S = t$, and $F = H$ we obtain

$$H = \frac{y^2}{2a(\lambda)} + b(\lambda)|x| + \varepsilon H_1(|x|, y, \lambda, \varepsilon). \tag{22}$$

Calculating the action as in the previous section

$$I = \frac{2\sqrt{2a}}{3b} H^{3/2},$$

we obtain

$$H_0(I, \lambda) = \frac{1}{2}\left[\frac{3b(\lambda)}{\sqrt{a(\lambda)}} I\right].$$

Applying the theorem we obtain a large set of caustics near the boundary for sufficiently large energy, provided $b(\lambda) > 0$.

2.2. Caustics in dual billiards. Dual billiard is a dynamical system defined in the exterior X of a convex closed oriented curve Γ in the plane. If $x \in X$ then $P(x) = y$, where $[x, y]$ is tangent to Γ at a point O, oriented as the curve, and $|x, O| = |O, y|$.

The stability problem for this system was formulated in [25] and studied later in [6, 10, 11, 30], see also a recent survey by Tabachnikov [30] for more references. We use a recent result by Boyland [6], where the dual billiard map was shown to be equivalent to an impact oscillator with the Hamiltonian given by

$$K = \frac{p^2}{2} + \frac{q^2}{2} + \rho(t)|q|,$$

where $\rho(t)$ is the curvature radius of the billiard boundary. We apply the theorem to show the existence of invariant curves in the small amplitude limit which corresponds to the caustics near the boundary. Indeed, rescaling the system

$$K = \varepsilon^2 F, \quad p = \varepsilon P,$$
$$s = \varepsilon S, \quad q = \varepsilon^2 Q,$$

we obtain the new Hamiltonian

$$F = \frac{P^2}{2} + \rho(\varepsilon S)|Q| + \varepsilon^2 \frac{Q^2}{2},$$

which satisfies the conditions of the theorem, provided $\rho(s) > 0$ and analytic.

2.3. Fermi–Ulam problem. The problem of stability of a ball bouncing elastically between two walls, one at rest the other one oscillating periodically, has been introduced by Fermi in order to explain the origin of the high-energy cosmic radiation [9]. It was further developed by Ulam [31] and others [8,29,18,21].

We slightly generalize the problem by assuming additional analytic time-dependent potential field. Therefore, in our problem the particle travels between two walls; one at $x = 0$, the other at $x = p(t)$ according to

$$\ddot{x} + V'(x, t) = 0.$$

Now, we use the transformation stopping the wall which originated in the theory of the heat equation to solve the free boundary problems and was later used for the quantum Fermi–Ulam problem, see [29] and references therein.

Introducing the new variable and the new time

$$x = p(t)y, \quad \tau = \int_0^t \frac{ds}{p^2(s)},$$

we obtain a system of a ball bouncing elastically between two walls at $y = 0$ and $y = 1$ and moving according to

$$y'' + p^3(t)V_x(p(t)y, t) + \ddot{p}(t)p^3(t)y = 0,$$

where $t = t(\tau)$. The new system is also periodic in τ with the period $T_\tau = \int_0^T \frac{dt}{p^2(t)}$. We again slightly generalize the problem by considering a particle bouncing elastically between the two stationary walls in an arbitrary analytic potential

$$y'' + W_y(y, \tau) = 0.$$

The Hamiltonian of the problem is given by

$$K = \frac{p_y^2}{2} + W(y, \tau).$$

This is equivalent to the system of a particle moving on the circle in a potential non-smooth at two points. If we let y be the angular variable $y \in (-1, 1)$ then the Hamiltonian takes the form

$$K = \frac{p_y^2}{2} + W(|y|, \tau).$$

Note that the exactness condition [19]

$$\oint p_y \, dy = \text{constant}$$

is satisfied because W is periodic in y. Rescaling and introducing more appropriate variables

$$p_y = \varepsilon^{-1} I, \quad y = \phi,$$
$$\tau = \varepsilon T, \quad K = \varepsilon^{-2} H,$$

we obtain

$$H = \frac{I^2}{2} + \varepsilon^2 W(|\phi|, \varepsilon T).$$

This system is already in the action-angle variables and $\omega = I \neq 0$, $\omega_I = 1 \neq 0$. Proceeding as in the proof of the theorem, we obtain the exact map which satisfies the conditions of the monotone twist theorem and therefore possess invariant curves. This implies the stability result for Fermi–Ulam problem.

3. Example of Instability

The billiard dynamics may be ill-defined even in a convex billiard with a continuous non-vanishing curvature. More precisely, Halpern constructed an example of a classical billiard with these properties, yet, possessing a trajectory reaching the boundary in finite time [12]. We present an analog example for the bouncing ball problem in the framework of smooth Hamiltonian systems and indicate how the construction can be carried over to the classical and dual billiards. This approach indicates the connection with the phenomenon of blow-up in finite-time in ODEs [7]. We also prove that the billiard flow is well defined for all time if the curvature $k > k_0 > 0$ and it is of bounded variation.

We start by constructing a piecewise constant $k(t) : 0.5 < k(t) < 1.5$, such that the equation $\ddot{x} + k(t)\mathrm{sgn}(x) = 0$ will have a solution coming to rest ($\dot{x} = 0, x = 0$) in finite time and then show how $k(t)$ can be made continuous. First, we take a particular trajectory in the autonomous system $k = 1$ and modify k as follows: when the solution has the largest distance from the origin x, k is decreased by Δk so that the energy changes by $\Delta H = \Delta k|x|$ (since $\dot{H} = \dot{k}|x|$). We change k back to the original value $k = 1$ when the solution passes through $x = 0$ so that the energy remains the same at this moment. This procedure can be continued indefinitely. At the n^{th} step the energy decreases by $\Delta k_n x_n$, where $x_n = H_n/k$. Since $k = 1$ when the energy decreases, then $\Delta H_n = \Delta k_n H_n$. Recalling that $0.5 \leq k \leq 1.5$ we obtain the estimate on the period of one oscillation $T_n \leq C\sqrt{H_n}$.

Since we are constructing a solution which looses its energy in finite time, we let $H_n = \frac{1}{n^3}$ so that $\sum T_n$ converges. The corresponding Δk_n is given by $|\Delta k_n| = \frac{|\Delta H_n|}{H_n} \leq \frac{c}{n}$. Starting with sufficiently large n_0 so that $|\Delta k_n| \leq 0.5$ for $n \geq n_0$ we obtain the example.

We can now make k continuous by approximating it in the neighborhoods of the jumps with continuous linear functions. Repeating the construction we obtain an example of instability with continuous k and with the same estimates, since the neighborhoods where k is modified can be chosen arbitrarily small.

This construction and its continuous modification carries over to the classical and dual billiards without any changes. Indeed, we have already showed that the bouncing ball problem is asymptotically close to the classical billiard near the boundary.

Therefore, constructing the sequences $\{H_n\}$, $\{\Delta k_n\}$, $\{T_n\}$, we will make errors of order $o(H_n)$, $o(\Delta k_n)$, and $o(T_n)$, which can be checked by direct calculations. Therefore, we will obtain a similar example and its continuous modification. The same argument applies to the dual billiard problem, see also [6].

Note, that the constructed k has unbounded variation. This turns out to be a necessary condition for such a construction. We prove this statement for the classical billiard

Theorem 3.1. *In a convex classical billiard with the curvature $k \in BV(S^1)$ and $k > k_0 > 0$ there is no trajectory which can reach the boundary in finite time.*

Proof. The Hamiltonian function for the classical billiard

$$H = 1 - (1 - k(t)|x|)\sqrt{1 - y^2}$$

is chosen so that near the boundary $H \geq 0$ and $H = 0$ iff $x = 0$ and $y = 0$. Assuming $y \leq 0.5$, we obtain an estimate for the distance from the boundary

$$|x|_{\max} \leq \frac{2}{\sqrt{3}} \frac{H}{k_{\min}},$$

and since

$$\dot{H} = \dot{k}|x|\sqrt{1 - y^2},$$

we obtain the inequality

$$|\dot{H}| \leq |\dot{k}| \frac{2}{\sqrt{3}} \frac{H}{k_{\min}},$$

where $\dot{k}$ exists a.e., since it is the derivative of a function of bounded variation [28]. Integrating, we obtain the energy decay estimate,

$$\frac{H_T}{H_0} \geq \exp^{-C \int_0^T |\dot{k}(t)| dt} > 0,$$

since $\int_0^T |\dot{k}(t)| dt$ is bounded by the variation of k on $[0, T]$ [28]. Therefore, a trajectory cannot reach the boundary in finite time. □

Remark 3.1. Another property of the system, which is crucial for the above instability examples, is that the period of oscillation would vanish as the solution approaches the boundary. It is these two properties that have been used by Coffman and Ulrich in [7], where an unbounded solution on the finite interval $t \in [0, t^*)$ for the equation $\ddot{x} + a(t)x^3 = 0$ has been constructed with $a(t)$ having unbounded variation near t^*. In this system the period of oscillation decays to zero as a solution grows unbounded.

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