Instability in Fermi–Ulam 'ping-pong' problem

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Abstract. The motion of a classical particle bouncing elastically between two parallel walls, with one of the walls undergoing a periodic motion is considered. This problem, called Fermi–Ulam 'ping-pong', is known to possess only bounded solutions if the motion of the wall is sufficiently smooth $p(t) \in C^{4+\epsilon}$, where $p(t)$ is the position of the wall. It is shown that the stability result does not hold if $p(t)$ is just a continuous function by providing two examples of instability. The second example also answers the question posed in Levi M and Zehnder E (1995 Boundedness of solutions for quasiperiodic potentials *SIAM J. Math. Anal.* **26** 1233–56) about instability in the 'squash player's' problem. Both examples are constructed for an equivalent system with motionless walls. The reduced system is obtained using the transformation, developed in the heat equation theory to solve the moving boundary problem.

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1. Introduction

The question about the stability of a particle bouncing elastically between two parallel walls, with one of the walls undergoing a periodic motion is one of the basic stability problems in Hamiltonian dynamics. Indeed, it is one of the simplest systems, where the stability problem is already nontrivial. This system was introduced by Fermi in an attempt to explain the origin of the high-energy cosmic radiation [3] and considered later by Ulam [12] and others [2, 5, 9, 11]. For various modifications of this problem see [4] and references therein. While the generating function of 1-period map can be explicitly written, see [5], its closeness to an integrable map becomes evident only after a few canonical transformations are carried out and near integrability can be established provided $p(t) \in C^{4+\epsilon}$, where $p(t)$ is the position of the wall [5]. The results rely on Moser's small twist theorem [8, 10] which requires that $p(t)$ has at least four derivatives. One might think that this smoothness requirement is due to the technical limitations rather than to the nature of the problem. We show that this is not the case by providing two examples with the particle accelerating to infinity with $p(t) \in C^0$.

The first example is very easy to construct and it gives explicitly a 1-parameter family of unbounded solutions. This is because the system with piecewise linear p turns out to be 'integrable'.

The second example is less explicit but it is more generic and more intuitive. It also answers the question posed by Zehnder and Levi [7] about instability in the 'squash player's' problem.

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2. Constructions

First, we carry out the transformation which will bring our system to a more convenient form for the constructions of unbounded solutions. In the next two sections we present two examples based on the system obtained in the first section.

2.1. The transformation 'stopping' the wall

This transformation has appeared in the heat equation theory to solve the moving boundary problems and it was later used for the quantum 'Fermi–Ulam' problem, see [11] and references therein. It is closely related to the Liouville transformation, see chapter 6.3 in [1].

Consider a particle bouncing between a stationary wall $x = 0$ and a periodically moving 'paddle' $x = p(t)$. Between the collisions the particle is free $\ddot{x} = 0$. Let the new position variable be $y = x/p(t)$, where p is periodic and positive, then the new equation of motion, away from the walls, is given by

$$\ddot{x} = \ddot{p}y + 2\dot{p}\dot{y} + p\ddot{y} = 0.$$

In order to remove the $\dot{y}$ term we choose the new time $\tau = f(t)$ so that the equation takes the form

$$\ddot{p}y + 2\dot{p}\dot{f}y' + p\ddot{f}y' + p\dot{f}^2y'' = 0,$$

where $F' = \frac{\mathrm{d}}{\mathrm{d}\tau}F$. Requiring that $2\dot{p}\dot{f} + p\ddot{f} = 0 \Leftarrow p^2\dot{f} = 1$ we obtain

$$\ddot{p}y + p\dot{f}^2y'' = 0,$$

which is equivalent to

$$y'' + \ddot{p}p^3y = 0 \tag{1}$$

with $\tau = f(t) = \int_0^t \frac{\mathrm{d}s}{p^2(s)}$.

Now we ensure that both walls are indeed at rest by showing that for any t, where $\dot{p}$ is defined, the velocity of the particle changes only the direction but has the same magnitude. The subscripts $-$ and $+$ correspond to the moments just before and after the collision, respectively. For the wall at $x = 0$ we have:

$$\dot{x}_+ + \dot{x}_- = 0 \Leftrightarrow \dot{p}_+y_+ + p_+\dot{y}_+ + \dot{p}_-y_- + p_-\dot{y}_- = 0 \Leftrightarrow \frac{1}{p}(y'_+ + y'_-) = 0, \tag{2}$$

where we have used that $y_+ = y_- = 0$, $p_+ = p_- = p$, and $\dot{p}_+ = \dot{p}_- = \dot{p}$.

For the wall at $x = p(t)$ we have:

$$\dot{x}_+ + \dot{x}_- = 2\dot{p} \Leftrightarrow \dot{p}_+y_+ + p_+\dot{y}_+ + \dot{p}_-y_- + p_-\dot{y}_- = 2\dot{p} \Leftrightarrow \frac{1}{p}(y'_+ + y'_-) = 0, \tag{3}$$

where we have used that $y_+ = y_- = 1$, $p_+ = p_- = p$, and $\dot{p}_+ = \dot{p}_- = \dot{p}$.

Remark 2.1.1. We have obtained an equivalent system where the particle moves in the field of time-dependent quadratic potential $V = \frac{1}{2}\ddot{p}p^3y^2$ between two motionless walls at $p = 0$ and $p = 1$.

Remark 2.1.2. The evolution of the system is not well defined if the particle collides with the wall at the moment of discontinuity of $\dot{p}$. The corresponding initial conditions form a subset of zero measure. Below we will exclude this set from our consideration.

In the next two sections we will construct examples of instability for this system.

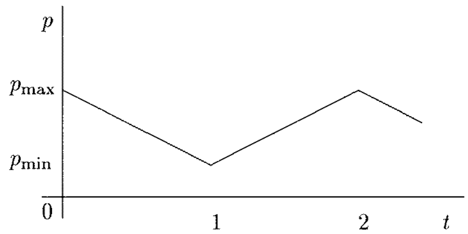

Figure 1. Motion of the wall in the first example of instability.

2.2. Example of instability with piecewise constant velocity of the wall

We will use the equivalent description of the original system obtained in the previous section to construct an example of instability with piecewise linear p. This choice of p is quite natural as the forcing term $\ddot{p}p^3y$ in (1) is a sum of δ-functions and therefore equation (1) can be explicitly integrated.

First, we calculate the change of the velocity at the moment when $\dot{p}$ is discontinuous, which will be used in the constructions. Integrating equation (1) we obtain

$$y'_+ - y'_- = -\int_{\tau_-}^{\tau_+} \ddot{p}p^3y\,\mathrm{d}\tau = -\int_{t_-}^{t_+} \ddot{p}p^3y\frac{\mathrm{d}\tau}{\mathrm{d}t}\,\mathrm{d}t = -\int_{t_-}^{t_+} \ddot{p}py\,\mathrm{d}t = -py\Delta\dot{p}. \tag{4}$$

Let $p(t) > 0$ be a 2-periodic function defined by (see figure 1)

$$\begin{cases} p(t) = p_{\mathrm{max}} - kt & \text{if } t \in (0, 1) \\ p(t) = p_{\mathrm{min}} + k(t-1) & \text{if } t \in (1, 2), \end{cases} \tag{5}$$

where $k = p_{\mathrm{max}} - p_{\mathrm{min}}$. We make a restriction on $p(t)$ requiring that

$$\int_0^2 \frac{\mathrm{d}s}{p^2(s)} = 2 \tag{6}$$

so that the new system is also 2-periodic in τ. Integrating (6) with $p(t)$ given by (5) we obtain the relation

$$p_{\mathrm{min}}p_{\mathrm{max}} = 1. \tag{7}$$

Due to the choice of a piecewise constant $\dot{p}$ and the convenient transformation, the particle's velocity changes only twice a period: increases at $\tau = 0$ and decreases at $\tau = 1$.

By our construction $|\Delta\dot{p}| = 2(p_{\mathrm{max}} - p_{\mathrm{min}})$ and therefore

$$\Delta y'(0) = 2yp_{\mathrm{max}}(p_{\mathrm{max}} - p_{\mathrm{min}})$$
$$\Delta y'(1) = -2yp_{\mathrm{min}}(p_{\mathrm{max}} - p_{\mathrm{min}}).$$

Choosing $p_{\mathrm{max}} = \sqrt{2}$ and $p_{\mathrm{min}} = 1/\sqrt{2}$ we obtain

$$\Delta y'_+ = 2y$$
$$\Delta y'_- = -y.$$

Now, we prove that there is no uniform bound on the velocity by constructing an unbounded solution. Using time periodicity of the system we consider the 1-period map P, which inherits the stability properties of the system and is composed of the four maps

$$P = P_4 \circ P_3 \circ P_2 \circ P_1, \tag{8}$$

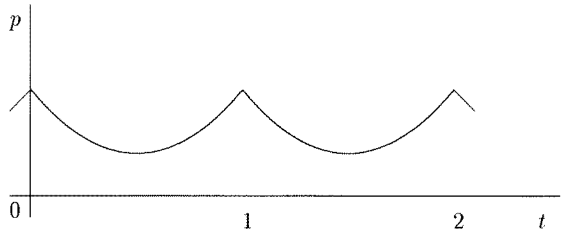

Figure 2. Wall motion corresponding to the 'squash player's' problem.

where

$$P_1(y, y') = (y, y' + 2y)$$
$$P_3(y, y') = (y, y' - y)$$

and $P_2 = P_4$ maps the current position of the particle into its position after $\Delta\tau = 1$ of free motion.

Consider the family of initial conditions given by $\tau = 0$, $y = 1-\lambda$, and $y' = 2N+1+\lambda$, where $\lambda \in (0, \frac{1}{2})$ and $N \in \mathbb{Z}^+$. We show that $P(y, y') = (y, y' + 2)$ for any $(y, y') = (1-\lambda, 2N+1+\lambda)$, i.e. this set is mapped into itself and its iterates are unbounded. Indeed, applying consequently the transformations we have

P_1: $(1 - \lambda, 2N + 1 + \lambda) \rightarrow (1 - \lambda, 2N + 1 + \lambda + 2 - 2\lambda) = (1 - \lambda, 2(N + 1) + 1 - \lambda)$

P_2: $(1 - \lambda, 2(N + 1) + 1 - \lambda) \rightarrow (2\lambda, -[2(N + 1) + 1 - \lambda])$

P_3: $(2\lambda, -[2(N + 1) + 1 - \lambda]) \rightarrow (2\lambda, -[2(N + 1) + 1 - \lambda] - 2\lambda) = (2\lambda, -[2(N + 1) +1 + \lambda])$

P_4: $(2\lambda, -[2(N + 1) + 1 + \lambda]) \rightarrow (1 - \lambda, 2(N + 1) + 1 + \lambda)$.

Remark 2.2.1. This map reduces to a linear automorphism on the 2-torus and its eigenvector corresponds to the above subset of initial conditions producing unbounded solutions.

Remark 2.2.2. The quantum version of the above example has been earlier constructed in [11].

2.3. Example of instability in the 'squash player's' problem

Now, we consider an example of instability with $p(t)$ which is no longer piecewise linear (see figure 2). It turns out that using the approach developed in [6] the existence of an unbounded solution can be established. The construction works with any $p(t)$ which is smooth on $t \in (0, 1)$ and has a sufficiently large jump in the derivative at $t = 0$. However, we consider specific $p(t)$ to make the estimates easier. We choose $p(t)$ which solves the differential equation $\ddot{p}p^3 = 1$

$$p(t) = \frac{1}{\sqrt{\alpha}}\sqrt{1 + \alpha^2(t - \tfrac{1}{2})^2} \qquad \text{for } t \in (0, 1) \tag{9}$$

so that the equation of motion (1) would become $y'' + y = 0$ away from the points of discontinuities of $\dot{p}$.

This problem also represents an example of the 'squash player's' problem where there are a few periodically moving 'walls' with positions $x = p_i(t) > 0$. The ball hits the closest wall between two successive collisions with the stationary wall at

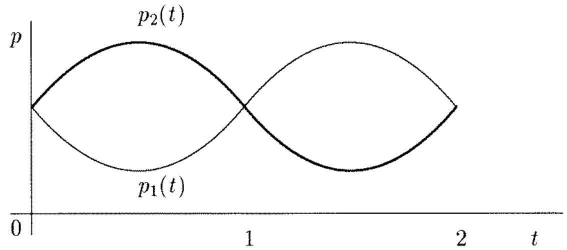

Figure 3. 'Squash player's' problem. The heavy curve $p_2(t)$ represents the motion of one wall, the light curve $p_1(t)$ represents the motion of the other wall.

$x = 0$. The question arises whether the velocity of the ball is uniformly bounded in time. The problem is equivalent to the 'Fermi–Ulam' problem with one moving wall at $x = p(t) = \min\{p_1(t), p_2(t), \ldots, p_n(t)\}$.

Choosing 2-periodic $p_1(t) \in C^\infty(0, 2)$ which coincides with (9) on $t \in (0, 1)$, and is convex up on $t \in (1, 2)$, and $p_2(t) = p_1(t + 1)$ (see figure 3) we then obtain $p(t) = \min\{p_1(t), p_2(t)\}$ coinciding with (9).

Now, we calculate the Poincaré map for the problem under consideration. After straightforward calculations we obtain

$$y'_+ - y'_- = py\Delta\dot{p} = y\alpha.$$

The new period is given by

$$T_\tau = \int_0^1 \frac{\mathrm{d}t}{p^2(t)} = 2\arctan\frac{\alpha}{2} \leqslant \pi.$$

The Poincaré map P is composed of the two maps

$$P_1 : y'_+ = y'_- + \alpha y$$

and T_τ-periodic map P_2 which gives the evolution of a particle bouncing between two walls located at $y = 0, 1$ and moving between the walls in quadratic potential

$$y'' + y = 0.$$

The phase space of this system can be represented as a cylinder with the angular variable $y \in (-1, 1)$ and the vertical coordinate $z = y' \in \mathbb{R}^+$, where $y \in [0, 1)$ corresponds to the motion with positive velocity and $y \in (-1, 0]$ to the motion with negative velocity (see figure 4). We will also use the covering space $(Y, Z) \in \mathbb{R} \times \mathbb{R}^+$.

For large velocities the map P_2 is well approximated by the map corresponding to $y'' = 0$. Indeed, using the energy identity

$$\frac{z^2}{2} + \frac{y^2}{2} = \frac{z_0^2}{2} + \frac{y_0^2}{2}$$

and $|y| \leqslant 1$ we obtain

$$|z^2 - z_0^2| \leqslant 1 \Leftrightarrow |z - z_0| \leqslant \frac{1}{z_0} \qquad \text{if } |z_0| \gg 1, \tag{10}$$

and thus,

$$P_2(Y, Z) = \left(Y + T_\tau Z + \mathrm{O}\left(\frac{1}{Z}\right), Z + \mathrm{O}\left(\frac{1}{Z}\right)\right). \tag{11}$$

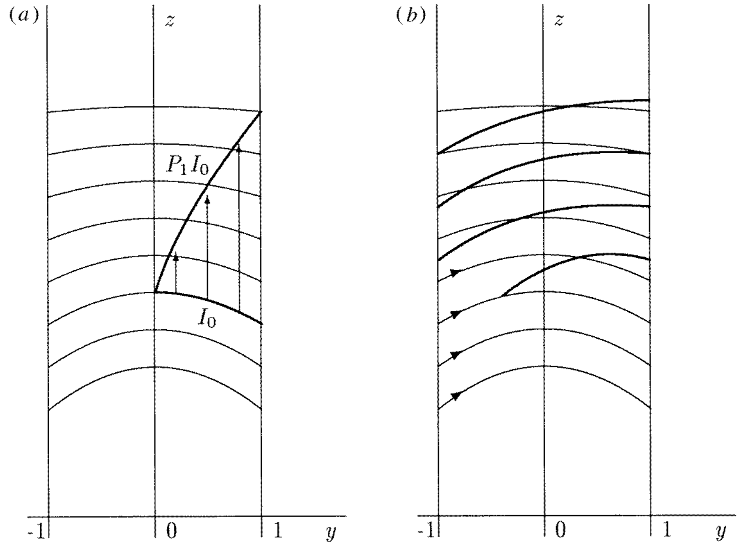

Figure 4. One step of the construction of an unbounded solution. The light curves are the integral lines of $y'' + y = 0$ and the heavy curves are the iterates of I_0. (a) I_0 and $P_1 I_0$, (b) $PI_0 = P_2 \circ P_1 I_0$.

Now, we describe the construction, which is similar to the one in [6] (see figure 4). First, we choose the set I_0 which coincides with the positive part of a phase trajectory of P_2, with $y \in (0, 1)$ and z sufficiently large. Under the action of P_1 the initial curve will shear upwards as shown in figure 4(a). Under the action of P_2 the curve will spiral round the cylinder (figure 4(b)). Choosing a subcurve I_1 from $PI_0 = P_2 \circ P_1 I_0$, where $y \in (0, 1)$ and $\min[z(I_1)]$ is sufficiently larger than $\max[z(I_0)]$, we repeat the process. In this way we obtain a sequence of enclosed curves

$$I_0 \supset P^{-1} I_1 \supset P^{-2} I_2, \ldots,$$

which by the lemma on enclosed intervals contains at least one point belonging to all the curves and having unbounded iterates by the construction.

We justify the above construction by proving the inductive lemma.

Lemma 2.1. *Let C^1_M denote the space of curves $z = v(y) > M$ defined on $y \in (0, 1)$ and satisfying the steepness condition $\frac{\mathrm{d}v(y)}{\mathrm{d}y} > -\frac{y}{v(y)}$, i.e. it is steeper than the vector field of $y'' + y = 0$. Then for any $v_0(y) \in C^1_M$ its image $P(y, z = v_0(y))$ contains $v_1(y) \in C^1_M$ such that $v_1(0) > v_0(0) + 1$, provided M and α are sufficiently large.*

Proof. Let M be so large that the error terms in (11) are smaller than 0.01 and let $\alpha > 10$ and such that $T_\tau \geqslant 1$. Consider $P_1(y, v_0(y)) = (y, v_0(y) + \alpha y)$ for $y \in (\frac{1}{2}, 1)$. Because of (10) and due to the choice of M we have $v_0(1) - v_0(\frac{1}{2}) > -0.01$ and $v_0(\frac{1}{2}) - v_0(0) > -0.01$. Therefore $\Pi_Z P_1(1, v_0(1)) - \Pi_z P_1(1/2, v_0(\frac{1}{2})) \geqslant 5 - 0.01 \geqslant 4$ and $\Pi_Z P_1(y, v_0(y))_{y \in (\frac{1}{2}, 1)} - v_0(0) \geqslant 5 - 0.01 \geqslant 4$, where Π_Y and Π_Z denote projections onto Y and Z axes, respectively. Applying P_2 to $P_1(y, v_0(y))_{y \in (1/2, 1)}$ we obtain

(1) $\Pi_Y P(1, v_0(1)) - \Pi_Y P(\frac{1}{2}, v_0(\frac{1}{2})) \geqslant 4 - 0.01 \times 2 \geqslant 3$

(2) $\Pi_Z P(y, v_0(y))_{y \in (\frac{1}{2}, 1)} \geqslant v_0(0) + 4 - 0.01 \geqslant v_0(0) + 1$.

Due to the first inequality the image, $P(y, v_0(y))_{y\in(1/2,1)}$ makes at least one loop round the cylinder and therefore there exists a subcurve $(y, v_1(y))_{y\in(0,1)}$, which by the second inequality satisfies the property $v_1(0) \geqslant v_0(0)+1$. It also belongs to C^1_M since P_1 increases the slope of a curve on $y \in (0, 1)$ and P_2 cannot make the slope smaller than that of the vector field since P_2 is generated by the same vector field. □

Acknowledgments

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