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A note on adiabatic invariance in Hamiltonian systems depending singularly on the slow time

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Abstract

A one-degree-of-freedom oscillatory Hamiltonian system with a parameter depending singularly on the slow time is considered. It is shown that the system possesses an adiabatic invariant and its asymptotics is estimated for a rather general type of singularity. The leading term of the asymptotics turns out to be given by an integral of Fresnel type and the order of asymptotics is related to the type of singularity (stronger singularities cause larger change of adiabatic invariant). The result is applied to estimate the reflection coefficient in the problem of scattering electromagnetic wave on an obstacle with refraction index depending singularly on the coordinate. The relation to the stationary phase method is outlined. © 1998 Elsevier Science B.V.

1. Introduction

The notion of adiabatic invariant was introduced at the beginning of the century by Lorentz and Einstein in order to explain conservation of the ratio of energy to the frequency of radiation of an atom. As a model problem they considered one-degree-of-freedom Hamiltonian system with a parameter depending weakly on time. The energy of a such system may change considerably in the course of evolution. However, due to adiabaticity of the system, the energy is approximately some function of the parameter. Because of this relation, some quantity, called adiabatic invariant, remains almost constant for a long time. One of the simplest systems of this type is a harmonic oscillator whose frequency changes slowly with time. The adiabatic invariant here is the ratio of the energy and the frequency.

The majority of works on adiabatic invariance deal with the smooth parameter dependence on the slow time, see [1,8,10–13], since such systems are frequently encountered in applications, see [4,5,7]. Nevertheless, the case of singular dependence also occurs. For example in [14] an external periodic forcing with $\sqrt{\epsilon t}$-singularity was used to accelerate a classical particle in a potential with superquadratic growth at infinity.

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In this note, the case of quite general singular dependence of the parameter on the slow time is considered. Specifically, we investigate the behavior of the system in a neighborhood of one singularity. Without loss of generality, we assume that $p(\lambda)$ has a singularity at $\lambda = 0$ and $p(0) = 0$. We also assume that $p(\lambda)$ is sufficiently smooth $p \in C^2(0, \infty)$ and constant at infinity $p(\lambda) = p_+$ if $\lambda \geq \lambda_+$.

The Hamiltonian of the problem is given by

$$H = H(x, y, p(\epsilon t))$$

and the corresponding equations of motion take the form

$$\dot{x} = \frac{\partial H}{\partial y}(x, y, p), \qquad \dot{y} = -\frac{\partial H}{\partial x}(x, y, p). \tag{1}$$

Since the system is oscillatory, then in the phase space (x, y) of the unperturbed problem there is an open domain containing only closed trajectories for each value of the parameter. In this domain there can be defined action–angle variables for the autonomous system [9]. The action variable $I(x, y, p)$ is the area bounded by the trajectory passing through (x, y, p). The angle variable $\phi(x, y, p)$ is a uniformly changing angular variable along the trajectory of the autonomous system with the rate of change equal to the frequency. The transformation of variables, defined above, turns out to be canonical and the new Hamiltonian takes the form

$$H = H_0(I, p(\epsilon t)) + \epsilon p'(\epsilon t) H_1(I, \phi, p(\epsilon t)), \tag{2}$$

where H_0 is the original Hamiltonian expressed in the new variables and H_1 is the derivative of a generating function of the transformation to the action–angle variable with respect to the parameter, see [9] for the details. The equations of motion are given by

$$\dot{\phi} = \omega(I, p) + \epsilon p' \frac{\partial H_1}{\partial I}(I, \phi, p), \qquad \dot{I} = -\epsilon p' \frac{\partial H_1}{\partial \phi}(I, \phi, p), \tag{3}$$

where $\omega(I, p) = \partial_I H_0(I, p)$ is the frequency in the corresponding autonomous system.

2. Results

Since $p(\lambda)$ is twice continuously differentiable outside of the singularity, it is monotone and convex in some neighborhood of 0. By adiabatic invariance theorem the change of the action away from the singularity, where $|p'| \leq C$, is of order ϵ. Therefore, by assuming that $p(\lambda)$ is convex and monotone on $\lambda \in (0, \lambda_+)$ and constant on $\lambda \in (\lambda_+, \infty)$, see Fig. 1, we make an error of order ϵ, which is smaller than the change of the action in a neighborhood of singularity. For the above system we prove the following result.

Theorem 2.1. The asymptotics of change of the action variable on the interval $(0, \lambda_+ > 0)$ is given by an integral of Fresnel type [2]

$$\Delta I(\phi_0) = -\int_0^\infty \dot{p}(\epsilon t) \frac{\partial H_1}{\partial \phi}(I_0, \phi_0 + \omega_0 t, p_0)\, dt, \tag{4}$$

provided the Hamiltonian function is real analytic in x, y, p, the frequency does not vanish in the above domain, and p satisfies the following conditions:

[2] If $p = \sqrt{\epsilon t}$ and $H_1 = I \cos\phi$ then it is exactly Fresnel integral.

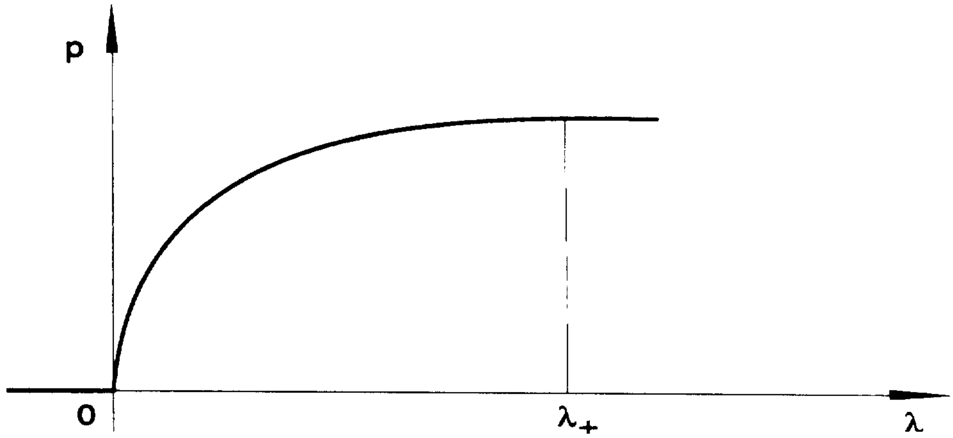

Fig. 1. Dependence of the parameter on the slow time.

1. There exists a function $\alpha(\epsilon)$, $0 < \epsilon \ll \alpha(\epsilon)$, such that

$$\left| \int_0^\infty \dot{p}(\epsilon t) \frac{\partial H_1}{\partial \phi}(I_0, \phi_0 + \omega_0 t, p_0)\, \mathrm{d}t \right| \le C\alpha(\epsilon), \tag{5}$$

$$\left| \int_0^\infty \dot{p}(\epsilon t) \mathrm{e}^{\mathrm{int}}\, \mathrm{d}t \right| \ge c_n \alpha(\epsilon) \quad \text{for any } n \in \mathbb{Z} \setminus 0. \tag{6}$$

2. There exists $t_c(\epsilon) \gg 1$ such that

$$\dot{p}(\epsilon t_c) = \mathrm{o}(\alpha(\epsilon)), \qquad p^2(\epsilon t_c) t_c = \mathrm{o}(\alpha(\epsilon)). \tag{7}$$

Lemma 2.2. If $\alpha(\epsilon) = p(\epsilon)$ then (5) is always true.

Proof. Due to convexity and monotonicity of $p(\lambda)$ we have

$$1 \le \frac{p(b\epsilon)}{p(\epsilon)} \le b \quad \text{if } b > 1, \qquad 1 \ge \frac{p(b\epsilon)}{p(\epsilon)} \ge b \quad \text{if } b < 1, \tag{8}$$

for any $b > 0$. Using these inequalities we estimate the integral in (4) from above on the two intervals

$$\int_0^\infty \dot{p} \frac{\partial H_1}{\partial \phi}\, \mathrm{d}t = \int_0^a \dot{p} \frac{\partial H_1}{\partial \phi}\, \mathrm{d}t + \int_a^\infty \dot{p} \frac{\partial H_1}{\partial \phi}\, \mathrm{d}t.$$

The first integral is bounded by $Cp(a\epsilon)$. Using integration by parts, we estimate the second integral[3]

$$\int_a^\infty \dot{p} \frac{\partial H_1}{\partial \phi}\, \mathrm{d}t = \frac{1}{\omega_0} \dot{p} H_1 |_a^\infty - \frac{1}{\omega_0} \int_a^\infty \ddot{p} H_1\, \mathrm{d}t \le C\dot{p}(a\epsilon).$$

[3] In this paper C and c denote different positive constants which do not depend on ϵ.

Using convexity and monotonicity of $p(\lambda)$ we have $\dot{p}(a\epsilon) \leq p(a\epsilon)/a$. Applying (8): $p(a\epsilon) \in (p(\epsilon), ap(\epsilon))$, we obtain the upper bound on the integral $Cp(\epsilon)$. □

Remark 2.3. It is easy to check for $p(\lambda) = \lambda^{\alpha}$, where $0 < \alpha < 1$, that conditions (6) and (7) hold. Therefore, using Lemma 2.2, we can apply Theorem 2.1 to obtain the asymptotics (4) for the singularities of power type.

Lemma 2.4. Theorem 2.1 applies in the case of a logarithmic singularity $p(\lambda) = -1/\log\lambda$, which is stronger than any singularity of power type.

Proof. Conditions (7) can be easily checked for $t_c = 1$ with $\alpha(\epsilon) = p(\epsilon)$. Now we will show that

$$\left|\int_0^\infty \dot{p}(\epsilon t) e^{\mathrm{int}}\, dt\right| \geq c_n p(\epsilon) \quad \text{for any } n \in \mathbb{Z} \setminus 0,$$

which will imply the result by Lemma 2.2. The above inequality is true if it is true for

$$\int_0^\infty \dot{p}(\epsilon t)\cos nt\, dt = \frac{1}{n}\int_0^\infty \dot{p}(\epsilon_n\tau)\cos\tau\, d\tau,$$

where $\epsilon_n = \epsilon/n$. We estimate the last integral from below using the series representation

$$\int_0^\infty \dot{p}(\epsilon_n\tau)\cos\tau\, d\tau = \sum_{m=0}^\infty (-1)^m a_m, \tag{9}$$

where

$$a_0 = \left|\int_0^{\pi/2} \dot{p}(\epsilon_n\tau)\cos\tau\, d\tau\right|, \qquad a_m = \left|\int_{\pi(2m-1)/2}^{\pi(2m+1)/2} \dot{p}(\epsilon_n\tau)\cos\tau\, d\tau\right| \quad \text{if } m \geq 1 \tag{10}$$

and making the following observations:

1. The series (9) has alternating terms. This follows from $\dot{p} \geq 0$ and properties of $\cos\tau$.
2. The terms a_m do not increase and $a_0 \gg a_1$.

The first observation implies

$$\sum_{m=2}^\infty (-1)^m a_m \geq 0 \Rightarrow \sum_{m=0}^\infty (-1)^m a_m \geq a_0 - a_1.$$

Therefore, using $a_0 \gg a_1$, we have1

$$\sum_{m=0}^\infty (-1)^m a_m \geq a_0 - a_1 = a_0 + o(a_0).$$

Now it remains to estimate a_0 from below and a_1 from above

$$a_0 = \int_0^{\pi/2} \dot{p}(\epsilon_n \tau) \cos\tau \, d\tau \geq \int_0^{\pi/4} \frac{\sqrt{2}}{2} \dot{p}(\epsilon_n \tau) = \frac{\sqrt{2}}{2} p\left(\epsilon_n \frac{\pi}{4}\right) \geq -\frac{c_n}{\log\epsilon},$$

$$a_1 = \left| \int_{\pi/2}^{3\pi/2} \dot{p}\,(\epsilon_n \tau) \cos\tau \, d\tau \right| \leq p\left(\epsilon_n \frac{\pi}{2}\right) - p\left(\epsilon_n \frac{3\pi}{2}\right) \leq -\frac{C_n}{\log^2\epsilon},$$

if ϵ is sufficiently small. □

Remark 2.5. The proof of Theorem 2.1 is based on the estimation of the evolution of the action variable on two intervals $(0, \lambda_c) \cup (\lambda_c, \lambda_+) = (0, \lambda_+)$, where λ_c is to be defined in the proof. The first interval is taken so small that the difference between the change of the action variable on the real solution and on the approximate solution, generated by an autonomous system, is smaller than the change itself. The change of the action variable on this interval will contain the leading term of the asymptotics. On the second interval the parameter evolves so slowly that the change of the action is much smaller than the leading term of the asymptotics.

Remark 2.6. The estimation of the integral in (4) is related to the stationary phase method. Indeed, choosing p as a new variable of integration we obtain

$$\Delta I(\phi_0) = -\int_0^{p_+} \frac{\partial H_1}{\partial \phi}\left(I_0, \phi_0 + \omega_0 \frac{\lambda(p)}{\epsilon}, p_0\right) dp,$$

where $\lambda(p)$ is the inverse function of $p(\lambda)$ and therefore $\lambda'(0) = 0$.

Remark 2.7. There are singularities which violate condition (7) for any choice of t_c, see Appendix B for an example.

3. Proofs

Lemma 3.1. Under the conditions of Theorem 2.1, the asymptotics of the integral in (4) is generically of order $\alpha(\epsilon)$. More precisely for any $\nu > 0$ there exists $\delta > 0$ such that

$$\mu\{\phi_0 \in \mathbb{S}^1 \mid |\Delta I(\phi_0)| < \delta\alpha(\epsilon)\} < \nu.$$

Proof. Since the Hamiltonian in the action–angle variables is also real analytic, see [12], $f(\phi) = \partial H_1(I_0, \phi, p_0)/\partial\phi$ can be represented by convergent Fourier series

$$f(\phi) = \sum_{n \in \mathbb{Z}\backslash 0} f_n e^{in\phi}$$

in $\Pi_\rho = \{\phi \in \mathbb{C}^1 | \, |\mathrm{Im}\,\phi| \leq \rho\}$ for some $\rho > 0$ with exponentially decaying coefficients $|f_n| \leq \|f\|_\rho e^{-|n|\rho}$, where $\|f\|_\rho = \sup_{\phi \in \Pi_\rho} |f(\phi)|$.

Let $g(\phi_0)$ denote the integral in (4), then analyticity of $f(\phi)$ implies analyticity of $g(\phi_0) = \sum g_n e^{in\phi_0}$ in the same domain. Indeed, expanding g in Fourier series

$$g(\phi_0) = \int_0^\infty \dot{p}(\epsilon t) \frac{\partial H_1}{\partial \phi}(I_0, \phi_0 + \omega_0 t, p_0)\, dt = \int_0^\infty \dot{p}(\epsilon t) \sum_{n\in\mathbb{Z}\setminus 0} f_n e^{in(\phi_0+\omega_0 t)}\, dt$$
$$= \sum_{n\in\mathbb{Z}\setminus 0} f_n e^{in\phi_0} \int_0^\infty \dot{p}(\epsilon t) e^{in\omega_0 t}\, dt$$

we obtain $|g_n| \le p_+ |f_n| \le C e^{-|n|\rho}$. Applying condition (6) we have

$$|g_n| \ge c_n \alpha(\epsilon) |f_n|$$

and since for some N, $f_N \ne 0$ (for otherwise the system would be trivial) then $g_N \ge c\alpha(\epsilon)$.

Consider $h_\epsilon(\phi) = g(\phi)/\alpha(\epsilon)$ which is also an analytic function in Π_ρ, uniformly bounded $\|h_\epsilon(\phi)\|_\rho \le C$. Using the lower bound on g_N we also have $|h_N| \ge c > 0$. Now, using these observations, we will show that the relative measure of ϕ_0, where $|h(\phi_0)| \le \delta$ tends to 0 with $\delta \to 0$, which will prove the lemma.

Proposition 3.2. For any $\nu > 0$ there exists $\delta > 0$ such that $\mu\{\phi \in \mathbb{S}^1 \mid h(\phi) < \delta\} < \nu$ if h is analytic in Π_ρ, $\|h\|_\rho \le C$, and $|h_N| \ge c > 0$.

Proof. Suppose that the statement is wrong. Then there exists $\nu > 0$ and two sequences $\{\delta_n\}_{n=0}^\infty (\delta_n \to 0)$, $\{h_n\}_{n=0}^\infty$ such that $\mu\{\phi \in \mathbb{S}^1 \mid |h_n(\phi)| < \delta_n\} > \nu$, where h_n stands for a term of the sequence $\{h_n(\phi)\}_{n=0}^\infty$.

Since $\|h\|_\rho \le C$ we have $\|h'\|_{\rho-\sigma} \le C/\sigma$, see [3]. Fixing σ we obtain that $\{h_n\}$ is a uniformly bounded equicontinuous sequence. Therefore, by Ascoli-Arzela's lemma it contains a subsequence $\{h_{n_k}\}_{k=0}^\infty$ converging uniformly to a function h_0. Multiplying the difference $h_{n_k} - h_0$ with an exponent $e^{-ik\phi}$ and integrating over the period we obtain that all Fourier coefficients converge uniformly, too. This implies that Fourier coefficients of h_0 satisfy the same estimates and therefore $\|h_0\|_{\rho-\sigma} \le 4C\sigma^{-1}$, see [3]. In particular $h_0(\phi)$ is an analytic function in $\Pi_{\rho-\sigma}$ for any $\sigma > 0$ and its Fourier coefficient $|h_{0N}| \ge c > 0$.

Now, we will show that $\mu\{\phi \in \mathbb{S}^1 \mid h_0(\phi) = 0\} \ge \nu$. For any $\chi > 0$ we can find m such that $\|h_{n_m} - h_0\|_{\rho-\sigma} \le \chi$. Therefore we have $\mu\{\phi \in \mathbb{S}^1 \mid |h_0| \le \chi + \delta_{n_m}\} \ge \nu$. Since we can choose χ and δ_{n_m} arbitrarily small then h_0 has to be identically 0, which contradicts the condition $|h_{0N}| \ge c > 0$. □

Proof of Theorem 2.1 First, we consider the motion on the interval $0 \le t \le t_c$, where $0 \le t_c \ll 1/\epsilon$; the precise value of t_c will be specified later. Integrating (3) and using boundedness of the corresponding derivatives of H_1 in the interval $I_1 \le I \le I_2$ we obtain the following estimates:[4]

$$|I - I_0| \le Cp(\epsilon t), \qquad |\phi - \phi_0 - \omega(I_0, p_0)t| \le C(P(t) + p(\epsilon t)), \tag{11}$$

where

$$P(t) = \int_0^t p(\epsilon\tau)\, d\tau.$$

[4] A solution will not leave the interval (I_1, I_2), where the estimates hold, since, as we will see later, on the time-interval under consideration $\Delta I = o(1)$.

Integrating the equation for $\dot{I}$ in (3) and using the estimates in (11) we obtain

$$I_c = I_0 - \int_0^{t_c} \dot{p}\frac{\partial H_1}{\partial \phi}(I, \phi, p)\,\mathrm{d}t = I_0 - \int_0^{t_c} \dot{p}\frac{\partial H_1}{\partial \phi}(I_0, \phi_0 + \omega_0 t, p_0) + R, \tag{12}$$

where

$$\begin{aligned}
|R| &\leq \int_0^{t_c} |\dot{p}| \left| \frac{\partial H_1}{\partial \phi}(I, \phi, p) - \frac{\partial H_1}{\partial \phi}(I_0, \phi_0 + \omega_0 t, p_0) \right| \mathrm{d}t \\
&\leq \int_0^{t_c} C\dot{p}(p + P)\,\mathrm{d}t \\
&\leq C\left(p^2(\epsilon t_c) + \int_0^{t_c} \ddot{P} P\,\mathrm{d}t \right) \\
&\leq C\left(p^2(\epsilon t_c) + P(t_c)p(\epsilon t_c) + \int_0^{t_c} p^2\,\mathrm{d}t \right) \\
&\leq C(p^2(\epsilon t_c) + p^2(\epsilon t_c)t_c).
\end{aligned}$$

Since it is desirable that the change of the action in (12) contain the leading term of the asymptotics, we require $|R| \ll \alpha(\epsilon)$, which follows from condition (7)

$$p^2(\epsilon t_c)t_c \ll \alpha(\epsilon).$$

Now, we have to show that on $\lambda \in (\lambda_c, \lambda_+)$ the change of the action variable is much less than $\alpha(\epsilon)$. This will be done by applying a near-identical change of variables, which will absorb fast oscillations with the amplitude of order $\dot{p}$. As a result, the new action variable evolution will be easily estimated.

The symplectic change of variables can be defined implicitly by the generating function

$$S(J, \phi, t) = J\phi + \dot{p}W(J, \phi, p(\epsilon t))$$

as follows:

$$I = J + \dot{p}\frac{\partial W}{\partial \phi}(J, \phi, p(\epsilon t)), \qquad \psi = \phi + \dot{p}\frac{\partial W}{\partial J}(J, \phi, p(\epsilon t)).$$

The new Hamiltonian is given by

$$K(J, \psi, t) = H(I, \phi, p(\epsilon t)) + \frac{\partial S}{\partial t}(J, \phi, t),$$

where the old variables have to be expressed through the new variables via the above transformation. After substituting the expression for I, the Hamiltonian takes the form

$$K = H_0(J + \dot{p}W_\phi(J, \phi, p(\epsilon t)), p(\epsilon t)) + \dot{p}H_1(J + \dot{p}W_\phi(J, \phi, p(\epsilon t)), \phi, p(\epsilon t)) + \frac{\partial S}{\partial t}(J, \phi, t).$$

Using Taylor expansions we obtain

$$K = H_0(J, p(\epsilon t)) + \dot{p}\frac{\partial H_0}{\partial I}(J, p(\epsilon t))\frac{\partial W}{\partial \phi}(J, \phi, p(\epsilon t)) + \dot{p}H_1(J, \phi, p(\epsilon t)) + \mathrm{O}(\dot{p}^2) + \mathrm{O}(\ddot{p}).$$

We choose W so that to get rid of ϕ-dependent part in H_1

$$W = -\left(\frac{\partial H_0}{\partial I}(J, p(\epsilon t))\right)^{-1}\int_0^{\phi}\{H_1\}(J, \alpha, p(\epsilon t))\,\mathrm{d}\alpha,$$

where

$$\{H_1\} = H_1 - [H_1] \text{ and } [H_1] = \frac{1}{2\pi}\int_0^{2\pi} H_1(I, \phi, p(\epsilon t))\,\mathrm{d}\phi.$$

Then the new Hamiltonian takes the form

$$K = H_0(J, p(\epsilon t)) + \dot{p}[H_1](J, p(\epsilon t)) + \mathrm{O}(\dot{p}^2) + \mathrm{O}(\ddot{p}).$$

Thus, we obtain the estimate on the difference

$$I - J = \mathrm{O}(\dot{p}) \Rightarrow I - J = \mathrm{O}(\dot{p}(\lambda_c))$$

and on the rate of evolution of the new action variable

$$\dot{J} = \mathrm{O}(\dot{p}^2) + \mathrm{O}(\ddot{p}).$$

Integrating the last relation we obtain

$$J(\lambda_+) - J(\lambda_c) = \mathrm{O}(p(\lambda_+) - p(\lambda_c))\dot{p}(\lambda_c) + \mathrm{O}(\dot{p}(\lambda_c)) = \mathrm{O}(\dot{p}(\lambda_c)).$$

Using the above estimates we have

$$I(\lambda_+) - I(\lambda_c) = \mathrm{O}(\dot{p}(\lambda_c)).$$

Therefore, applying condition (7)

$$\dot{p}(\epsilon t_c) \ll \alpha(\epsilon)$$

we obtain

$$|I(\lambda_+) - I(\lambda_c)| \ll \alpha(\epsilon). \qquad \square$$

Appendix A. Scattering of electromagnetic waves

In this section, we apply the obtained result to calculate the asymptotics of reflection coefficient of scattering electromagnetic wave on an obstacle with a singular refraction index profile, see Fig. 2. The propagation of an electromagnetic wave in one dimension with frequency ω is governed by Helmholtz equation

$$E_{xx} + \omega^2 n^2(x) E = 0.$$

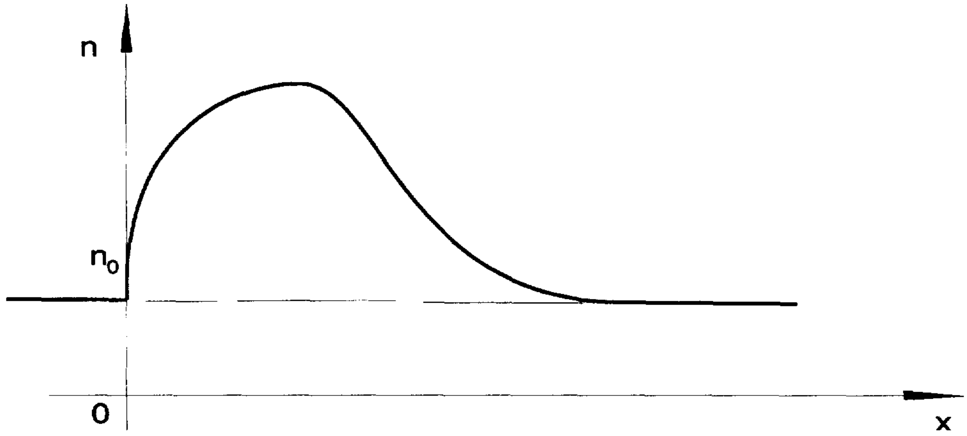

Fig. 2. The refraction index profile.

After rescaling the spatial variable $y = \omega x$ the equation takes the form

$$E_{yy} + n^2\left(\frac{y}{\omega}\right)E = 0. \tag{A.1}$$

Assuming that $n(y/\omega)$ satisfies the conditions of the theorem, we obtain complex adiabatic invariant for (A.1), whose real and imaginary parts are equal to the areas enclosed by the corresponding trajectories on $(\mathrm{Re}(E), \mathrm{Re}(E_y))$, $(\mathrm{Im}(E), \mathrm{Im}(E_y))$ planes.

Under the assumption that the wave comes from the left, there is only outgoing wave on the right-hand side given by

$$E = A\mathrm{e}^{\mathrm{i}n_0 y} \quad \text{for } y \to \infty,$$

while on the left-hand side there are both incident and reflected waves

$$E = \mathrm{e}^{\mathrm{i}n_0 y} + B\mathrm{e}^{-\mathrm{i}n_0 y} \quad \text{for } y \to -\infty.$$

It is well known that $|A|^2 + |B|^2 = 1$, see [6] for more details.

For $y \to -\infty$ we have

$$\mathrm{Re}(E) = (1 + \mathrm{Re}(B))\cos n_0 y + \mathrm{Im}(B)\sin n_0 y,$$

$$\mathrm{Re}(E_y) = -n_0(1 + \mathrm{Re}(B))\sin n_0 y + n_0 \mathrm{Im}(B)\cos n_0 y,$$

$$\mathrm{Re}(I) = \frac{\mathrm{Re}(E_y)^2}{2n_0} + \frac{n_0 \mathrm{Re}(E)^2}{2} = n_0((1 + \mathrm{Re}(B))^2 + \mathrm{Im}(B)^2),$$

$$\mathrm{Im}(E) = \mathrm{Im}(B)\cos n_0 y + (1 - \mathrm{Re}(B))\sin n_0 y,$$

$$\mathrm{Im}(E_y) = -n_0 \mathrm{Im}(B)\sin n_0 y + n_0(1 - \mathrm{Re}(B))\cos n_0 y,$$

$$\mathrm{Im}(I) = \frac{\mathrm{Im}(E_y)^2}{2n_0} + \frac{n_0 \mathrm{Im}(E)^2}{2} = n_0((1 - \mathrm{Re}(B))^2 + \mathrm{Im}(B)^2).$$

Due to the presence of only one exponent on the right-hand side, we obtain for $y \to +\infty$

$$\mathrm{Re}(I) = \mathrm{Im}(I) = n_0|A|^2.$$

Using the above equalities we have

$$\mathrm{Re}(\Delta I) = n_0|A|^2 - n_0(1 + 2\mathrm{Re}(B) + |B|^2),$$
$$\mathrm{Im}(\Delta I) = n_0|A|^2 - n_0(1 - 2\mathrm{Re}(B) + |B|^2).$$

Adding up the above equations we obtain

$$\frac{1}{2n_0}(\mathrm{Re}(\Delta I) + \mathrm{Im}(\Delta I)) = |A|^2 - |B|^2 - 1$$

and since $|A|^2 + |B|^2 = 1$ we obtain the asymptotic bound on the reflection coefficient

$$|B|^2 = -\frac{1}{4n_0}(\mathrm{Re}(\Delta I) + \mathrm{Im}(\Delta I)) = \mathrm{O}\left(\Delta n\left(\frac{1}{\omega}\right)\right).$$

Appendix B. Example of a function violating condition (7) of Theorem 2.1

We show that function

$$p(\lambda) = -\lambda \log \lambda$$

provides an example of monotone, convex function on the interval $(0, \mathrm{e}^{-2})$ having singularity at $\lambda = 0$ and, yet, violating condition (7) of Theorem 2.1. For $p = -\lambda \log \lambda$ condition (7) takes the form
(i) $\epsilon(-\log \epsilon t_c - 1) \ll -\epsilon \log \epsilon$,
(ii) $\epsilon^2 t_c^2 \log^2(\epsilon t_c) t_c \ll -\epsilon \log \epsilon$.
It is clear from the first inequality that t_c has to be larger than any power of ϵ, i.e. $t_c \geq \epsilon^{-\alpha}$ for any $\alpha > 0$. In the second inequality the expression on the left-hand side monotonically increases on the interval under consideration. Taking $t_c = \epsilon^{-2/3}$ we notice that it is already too large, therefore, there is no t_c satisfying both inequalities. In conclusion, we observe that even though the theorem does not apply in this case, it is not clear if the asymptotics will be different from (4).

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