Key
MATH 012 Final
No Graphing Calculator Portion

For the last time, I supply the reminder that I do not have graphing capabilities with this typesetter, so I will try to describe graphs to the best of my ability.

1. Graph the function $y = 3(x+3)^3 - 2$ using transformations, starting with the appropriate basic graph. Label each graph with the appropriate equation.

Solution: The first graph should be that of $y = x^3$. It is increasing everywhere and flattens out at $(0, 0)$. The next graph should be that of $y = (x+3)^3$, which is the previous graph shifted 3 to the left. The next graph should be that of $y = 3(x+3)^3$, which is the previous graph stretched away from the x axis by a factor of 3. The last graph should be that of $y = 3(x+3)^3 - 2$, which is the previous graph shifted down 2.

2. a. Determine the central axis, the amplitude, and the period of the function $f(x) = 2\sin(\pi x + \pi) - 5$.

Solution: The original equation can be rewritten as $f(x) = 2\sin[\pi(x+1)] - 5$.

central axis: $y = 5$

amplitude: 2

period: $\dfrac{2\pi}{\pi} = 2$

b. Graph two periods of the function $f(x) = 2\sin(\pi x + \pi) - 5$. One of the periods must include the y intercept.

Solution: By plugging in $x = 0$, one can obtain that the y intercept occurs at $(0, -5)$. The graph intersects the central axis exactly when x is an integer. The maximum value, which is -3, occurs at $x = \dfrac{-5}{2}$, $x = \dfrac{-1}{2}$, $x = \dfrac{3}{2}$, etc. The minimum value, which is -7, occurs at $x = \dfrac{-3}{2}$, $x = \dfrac{1}{2}$, $x = \dfrac{5}{2}$, etc.

3. a. Determine all intercepts and asymptotes of $y = \dfrac{x^2 - 5x + 6}{2x^2 - 14x + 12}$.

Solution: Note that $x^2 - 5x + 6$ factors as $(x-2)(x-3)$. Thus, the x intercepts occur at $(1, 0)$ and $(6, 0)$.

By plugging in $x = 0$, the y intercept of $\left(0, \dfrac{1}{2}\right)$ is obtained.

Note that $2x^2 - 14x + 12$ factors as $2(x-1)(x-6)$. Thus, the vertical asymptotes occur at $x = 1$ and $x = 6$.

Since the degrees of the numerator and denominator are equal, then there is a horizontal asymptote at $y = \dfrac{1}{2}$, and there is no slant asymptote.

b. Graph the function $y = \dfrac{x^2 - 5x + 6}{2x^2 - 14x + 12}$.

Solution: By using the sign line, one obtains that the function is positive when $x < 1$, negative when $1 < x < 2$, positive when $2 < x < 3$, negative when $3 < x < 6$, and positive when $x > 6$.

Since the function is positive when $x < 1$, then, as x approaches 1 from the left, y approaches ∞. Note that the y intercept occurs on the horizontal asymptote. Thus, the function is below the horizontal asymptote when $x < 0$ and approaches it as x approaches $-\infty$.

Since the function is negative when $1 < x < 2$, then, as x approaches 1 from the right, y approaches $-\infty$. Then the graph crosses through $(2, 0)$, is positive when $2 < x < 3$, and crosses through $(3, 0)$. Since the function is negative when $3 < x < 6$, then, as x approaches 6 from the left, y approaches $-\infty$.

Since the function is positive when $x > 6$, then, as x approaches 6 from the right, y approaches ∞. The graph approaches the horizontal asymptote as x approaches ∞.

4. a. Determine all intercepts and the vertex of $y = -2x^2 + 20x - 48$. State whether the vertex is a maximum or a minimum.

Solution: By plugging in $x = 0$, the y intercept of $(0, -48)$ is obtained.

One way to find the vertex and x intercepts (if they exist) is to put the equation into standard form.

$$\begin{aligned} y &= -2x^2 + 20x - 48 \\ &= -2(x^2 - 10x) - 48 \\ &= -2(x^2 - 10x + 25) - 48 + 50 \\ &= -2(x - 5)^2 + 2 \end{aligned}$$

Thus, the vertex is at $(5, 2)$. Since $a = -2$, the vertex is a maximum.

To determine the x intercepts, the above expression will be set equal to zero.

$$\begin{aligned} -2(x-5)^2 + 2 &= 0 \\ -2(x-5)^2 &= -2 \\ (x-5)^2 &= 1 \\ x - 5 &= \pm 1 \end{aligned}$$

$$\begin{array}{llcll} x - 5 & = 1 & \text{or} & x - 5 & = -1 \\ x & = 6 & \text{or} & x & = 4 \end{array}$$

Thus, the x intercepts are at $(6, 0)$ and $(4, 0)$.

b. Graph the function $y = -2x^2 + 20x - 48$. State the intervals on which this function is increasing and decreasing.

Solution: Hopefully, from the information in part a., the graph is obvious. The function is increasing on $(-\infty, 5)$ and is decreasing on $(5, \infty)$.

5. Graph the functions $y = \left(\frac{1}{3}\right)^x$ and $y = \log_{\frac{1}{3}} x$ on the same graph.

Solution: The graph of $y = \left(\frac{1}{3}\right)^x$ passes through the points $(-1, 3)$, $(0, 1)$, and $\left(1, \frac{1}{3}\right)$. This graph has a horizontal asymptote at $y = 0$.

The graph of $y = \log_{\frac{1}{3}} x$ passes through the points $\left(\frac{1}{3}, 1\right)$, $(1, 0)$, and $(3, -1)$. This graph has a vertical asymptote at $x = 0$.

6. Graph the circle $(x + 3)^2 + (y - 2)^2 = 16$.

Solution: The center of the circle is at $(-3, 2)$ and has radius 4. Thus, the graph of the circle passes through the points $(1, 2)$, $(-3, 6)$, $(-7, 2)$, and $(-3, -2)$.

7. Graph the complex number $5 - 3i$ and its complex conjugate on the same graph.

Solution: The horizontal and vertical axes of this graph should be labeled Re and Im, respectively. The complex number $5 - 3i$ corresponds to the point $(5, -3)$ on the Cartesian plane. Its complex conjugate $5 + 3i$ corresponds to the point $(5, 3)$ on the Cartesian plane.

8. a. Determine the line that passes through the point $(-1, 3)$ and is perpendicular to the line $y = \frac{1}{4}x + 2$.

Solution: The slope of the desired line is the opposite reciprocal of $\frac{1}{4}$, which is -4. Using the point-slope form of the equation of the line, the equation obtained is $y - 3 = -4(x - (-1))$. For part b., it is useful to have this line in slope intercept form. This is accomplished below.

$$\begin{aligned} y - 3 &= -4(x - (-1)) \\ y - 3 &= -4x - 4 \\ y &= -4x - 1 \end{aligned}$$

b. Graph the lines $y = \frac{1}{4}x + 2$ and the answer to part a. on the same graph.

Solution: The line $y = \frac{1}{4}x + 2$ has a y intercept of $(0, 2)$ and an x intercept of $(0, -8)$. The line $y = -4x - 1$ has a y intercept of $(0, -1)$ and an x intercept of $\left(\frac{-1}{4}, 0\right)$. Note that these lines intersect at the point $\left(\frac{-12}{17}, \frac{31}{17}\right)$. The angles formed by these lines should look like right angles, since they are supposed to be perpendicular.

9. Graph the polynomial function $y = -2(x+3)(x-2)^2(x+1)^5$ accurately. Label all intercepts.

Solution: The x intercepts occur at $(-3, 0)$, $(2, 0)$, and $(-1, 0)$. The y intercept occurs at $(0, -24)$.

Since the degree of the polynomial is 8 and the leading coefficient is negative, the end behavior of the graph is negative as x approaches either $-\infty$ or ∞.

The graph is negative when $x < -3$, crosses the x axis in a linear fashion at $x = -3$, is positive when $-3 < x < -1$, flattens as it crosses the x axis at $x = -1$, is negative when $-1 < x < 2$, crosses the y intercept of $(0, -24)$, touches the x axis at $x = 2$, and is negative when $x > 2$.

Since this is the last extra credit that deals with graphs, I will discuss it.

Extra Credit: Accurately graph the function $h(x) = \lfloor\, |\cos(\pi x)|\, \rfloor$. Determine a piecewise equation for the graph of this function. Show all work.

Solution: The graph of $y = \cos(\pi x)$ is relatively straightforward. Its central axis is at $y = 0$, its amplitude is 1, and its period is 2. Its x intercepts are exactly the points of the form $\left(\frac{k}{2}, 0\right)$ where k is an odd integer.

The graph of $y = |\cos(\pi x)|$ is obtained by reflecting all portions of the previous graph that are below the x axis across the x axis. Thus, this function achieves its maximum value of 1 whenever x is an integer, and it achieves its minimum value of 0 whenever $x = \frac{k}{2}$ for some odd integer k.

The value of $h(x) = \lfloor\, |\cos(\pi x)|\, \rfloor$ is 1 whenever x is an integer and 0 otherwise. Thus, a piecewise equation for it is

$$h(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z} \\ 0 & \text{if } x \notin \mathbb{Z}. \end{cases}$$